Bayesian approach to measuring parameter and model risk in loss ratio estimation

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(joint work with Hiroki KONDO)

1. Introduction

This talk is based on the joint work [3] with Hiroki Kondo of Nisshin Fire & Marine Insurance Company, Limited and of the Graduate School of Mathematics, Kyushu University.

It is a significant component of risk management for an insurance company to estimate the future loss ratio (the total losses divided by the total premiums) of a line of business, given the data for previous years. We shall focus on the problem of finding an estimator of the *Value-at-Risk* (VaR, quantile) of the future annual loss ratio, assuming that the annual loss ratios are independent and identically distributed. Here, for a positive number α slightly smaller than 1, the $100\alpha\%$ VaR of a random variable y is the value y_0 for which the probability that y is at most y_0 is α.

2. Conventional method

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be the given data of the annual loss ratios in the past n years. Write m_x and s_x^2 for its sample mean and (biased) sample variance respectively. The most conventional method assumes that the loss ratios have the normal distribution $N(\mu, \sigma^2)$, with its parameters estimated by the maximum likelihood estimators $\widehat{\mu} = m_x$ and $\widehat{\sigma} = s_x$. We then estimate the $100\alpha\%$ VaR of the future loss ratio by

$$m_x + z_\alpha s_x, \tag{1}$$

where z_α is the α-quantile of the standard normal distribution.

The method, however, has the major drawback that it takes into account only the *process risk* (caused by the stochastic nature of the model) and not the *parameter risk* (caused by the parameter estimation error) or the *model risk* (caused by using a wrong model). The next two sections present a Bayesian approach that gives an estimator of the VaR that takes all three types of risk into consideration.

3. Incorporating parameter risk

In what follows, with a slight abuse of notation, we shall not explicitly distinguish between random variables and their realizations; also, we shall always use the same letter f to denote probability density functions and probability mass functions of different random variables.

We assume that the annual loss ratios have the normal distribution $N(\mu, \tau^{-1})$ whose parameters (μ, τ) are again random variables on the parameter space $\mathbb{R} \times \mathbb{R}_{>0}$. Bayes' theorem tells us that the *posterior distribution* $f(\mu,\tau|\boldsymbol{x})$ is proportional to the product of the *prior distribution* $f(\mu,\tau)$ and the *likelihood*

$$f(\boldsymbol{x}|\mu,\tau)=\prod_{i=1}^{n} f(x_i|\mu,\tau)=\prod_{i=1}^{n}\sqrt{\frac{\tau}{2\pi}}\exp\biggl(-\frac{\tau(x_i-\mu)^2}{2}\biggr).$$

If we use, as the prior distribution on the parameter space $\mathbb{R}\times\mathbb{R}_{>0}$, the *normal-gamma distribution* $\mathrm{NG}(\alpha,\beta,\gamma,\delta)$ given by

$$f(\mu,\tau)=\sqrt{\frac{\delta}{2\pi}}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\tau^{\alpha-\frac{1}{2}}\exp\biggl(-\beta\tau-\frac{\delta\tau(\mu-\gamma)^2}{2}\biggr),$$

where $\alpha,\beta,\delta>0$ and $\gamma\in\mathbb{R}$ are parameters, then it turns out that the posterior distribution is also the normal-gamma distribution $\mathrm{NG}(\alpha',\beta',\gamma',\delta')$ with different parameters.

Following [2, Section 3.2], we shall use the improper prior distribution $f(\mu,\tau)\propto\tau^{-1}$ as the non-informative prior on the parameter space $\mathbb{R}\times\mathbb{R}_{>0}$. This improper prior can be thought of as $\mathrm{NG}(-1/2,0,0,0)$ and the posterior distribution turns out to be $\mathrm{NG}\bigl((n-1)/2,ns_x^2/2,m_x,n\bigr)$. Then the *posterior predictive distribution* of the future loss ratio y is given by

$$f(y|\boldsymbol{x})=\int_0^{\infty}\int_{-\infty}^{\infty}f(y|\mu,\tau)f(\mu,\tau|\boldsymbol{x})\,d\mu\,d\tau$$

and its $100\alpha\%$ VaR turns out to be

$$m_x+\sqrt{\frac{n+1}{n-1}}\,t_{\alpha}(n-1)s_x, \tag{2}$$

where $t_{\alpha}(n-1)$ is the α-quantile of Student's t-distribution with $n-1$ degrees of freedom.

4. Incorporating model risk

In order to find an estimator of the VaR with the model risk incorporated, we need to look at at least one model in which the loss ratio distribution is not normal. However, having used an improper prior $f(\mu,\tau)\propto\tau^{-1}$ in the normal model, we cannot compute the posterior distribution on the model space by the usual use of Bayes' theorem. The key observation to overcome this problem is that in the model where the loss ratios have the log-normal distribution, we can use exactly the same improper prior $f(\mu,\tau)\propto\tau^{-1}$ on the same parameter space $\mathbb{R}\times\mathbb{R}_{>0}$ as in the normal model. Write N and LN respectively for the normal and log-normal models.

Under the model LN, the estimators of the VaR corresponding to (1) and (2) are

$$\exp(m_{\log x}+z_{\alpha}s_{\log x}), \tag{3}$$

$$\exp\biggl(m_{\log x}+\sqrt{\frac{n+1}{n-1}}\,t_{\alpha}(n-1)s_{\log x}\biggr), \tag{4}$$

where $m_{\log x}$ and $s_{\log x}^2$ respectively denote the sample mean and the (biased) sample variance of $(\log x_1,\dots,\log x_n)$.

Partly inspired by [1], we now employ Bayesian inference for the expanded parameter space $\{\mathrm{N},\mathrm{LN}\}\times\mathbb{R}\times\mathbb{R}_{>0}$ equipped with the improper prior $f(M,\mu,\tau)\propto\tau^{-1}$. Then the posterior distribution turns out to be given by

- $f(\mathrm{N}\,|\boldsymbol{x})=p$ and $(\mu,\tau)|(\boldsymbol{x},\mathrm{N})\sim\mathrm{NG}\bigl((n-1)/2,ns_x^2/2,m_x,n\bigr)$;

- $f(\mathrm{LN}\,|\boldsymbol{x}) = 1 - p$ and $(\mu, \tau)|(\boldsymbol{x}, \mathrm{LN}) \sim \mathrm{NG}\big((n-1)/2, ns_{\log x}^2/2, m_{\log x}, n\big)$,

where

$$p = \frac{s_{\log x}^{n-1} \prod_{i=1}^n x_i}{s_{\log x}^{n-1} \prod_{i=1}^n x_i + s_x^{n-1}}.$$

It follows that the $100\alpha\%$ VaR of the posterior predictive distribution is given by the solution $q > 0$ to the equation

$$pF_{t(n-1)}\left(\frac{q - m_x}{\sqrt{(n+1)/(n-1)}\, s_x}\right) + (1-p)F_{t(n-1)}\left(\frac{\log q - m_{\log x}}{\sqrt{(n+1)/(n-1)}\, s_{\log x}}\right) = \alpha, \tag{5}$$

where $F_{t(n-1)}$ is the cumulative distribution function of Student's t-distribution with $n-1$ degrees of freedom.

5. Numerical example

If $\boldsymbol{x} = (0.33, 0.42, 0.37, 0.29, 0.31, 0.35, 0.42, 0.29, 0.23, 0.27)$ with $n = 10$, then the formulae (1)–(5) give the following estimators of the 99% VaR of the future loss ratio:

	(1)	(2)	(3)	(4)	(5)
distribution	N	N	LN	LN	N/LN
process risk	✓	✓	✓	✓	✓
parameter risk	✗	✓	✗	✓	✓
model risk	✗	✗	✗	✗	✓
estimated 99% VaR	0.466	0.513	0.494	0.571	0.558

Comparison among the estimators given by (1), (2), and (5) indicates how the estimated VaR increases as we incorporate the parameter and model risk one by one.

6. Concluding remarks

The Bayesian approach presented in this talk allows the VaR to be estimated computationally easily with both the parameter and model risk incorporated.

If we would like to use a different distribution as the loss ratio distribution, the parameter risk is often easy to incorporate into the estimated VaR but the model risk seems to be very difficult to incorporate because our method relies essentially on the fact that the *same* improper prior distribution can be used for both the normal and log-normal models. We probably need to find appropriate *proper* priors to handle other distributions.

References

[1] A. J. G. Cairns, *A discussion of parameter and model uncertainty in insurance*, Insurance Math. Econom. **27** (2000), 313–330.

[2] A. Gelman, J. B. Carlin, H. S. Stern, and D. B. Rubin, *Bayesian data analysis*, second ed., Texts in Statistical Science Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[3] H. Kondo and S. Saito, *Bayesian approach to measuring parameter and model risk in loss ratio estimation*, J. Math-for-Ind. **4** (2012), 85–89.