Properties of $\mathcal{G}_\delta$ Subsets of $\mathbb{R}$

Shingo SAITO*

Abstract

It is well-known that there exists a function from $\mathbb{R}$ to itself that is continuous on $\mathbb{R}\setminus\mathbb{Q}$ and is not continuous on $\mathbb{Q}$. In contrast, there is no function that is continuous on $\mathbb{Q}$ and is not continuous on $\mathbb{R} \setminus \mathbb{Q}$. We prove this by showing that the set of continuity points of a function is $\mathcal{G}_\delta$ and that a $\mathcal{G}_\delta$ set is completely metrizable. These, together with the Baire Category Theorem, imply the nonexistence of a function described above.

1 Complete Metrizability of Subsets of $\mathbb{R}$

Definition 1.1 *A subset of a topological space is said to be $\mathcal{G}_\delta$ if it is a countable intersection of open sets.*

Theorem 1.2 *A subset of a complete metric space is completely metrizable if and only if it is $\mathcal{G}_\delta$.*

Proof. We first prove the 'if' part. Let G be a $\mathcal{G}_\delta$ subset of a complete metric space (X, d) and take open sets G_1, G_2, $\ldots$ with $G = \bigcap_{j=1}^{\infty} G_j$. Define a map $f\colon G \longrightarrow X \times \mathbb{R}^{\mathbb{N}}$ by

$$f(x) = \left(x, \frac{1}{d(x, G_1^{\mathrm{c}})}, \frac{1}{d(x, G_2^{\mathrm{c}})}, \ldots\right)$$

We can easily see that f is continuous and injective. The inverse f^{-1} defined on $f(G)$ is also continuous because it is same as the restriction of the first projection $\pi_1\colon X \times \mathbb{R}^{\mathbb{N}} \longrightarrow G$ to $f(G)$. Therefore G is homeomorphic to $f(G)$.

Since $X \times \mathbb{R}^{\mathbb{N}}$ is a complete metric space, it suffices to prove that $f(G)$ is closed. Consider a sequence $\{f(x_n)\}$ in $f(G)$ converging to an element $(x, a_1, a_2, \ldots)$ in $X \times \mathbb{R}^{\mathbb{N}}$. Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, G_j^{\mathrm{c}})^{-1} = a_j$, we have $a_j = d(x, G_j^{\mathrm{c}})^{-1}$. It follows that $x \notin G_j^{\mathrm{c}}$ for every positive integer j, which implies $x \in G$. Thus $(x, a_1, a_2, \ldots) = f(x) \in f(G)$. Hence $f(G)$ is closed.

*`new@ms.u-tokyo.ac.jp`

Now we prove the 'only if' part. Let G be a completely metrizable subset of a complete metric space X. Let ρ be a complete metric on G compatible with its topology. Set $A = \{x \in \overline{G} \mid \inf \operatorname{diam}_\rho(G \cap U) = 0\}$, where the infimum is taken over all open neighbourhoods U of x. The set A is $\mathcal{G}_\delta$ in $\overline{G}$ because

$$\begin{aligned} A &= \{x \in \overline{G} \mid \inf \operatorname{diam}_\rho(G \cap U) = 0\} \\ &= \bigcap_{k=1}^{\infty} \left\{ x \in \overline{G} \;\middle|\; \inf \operatorname{diam}_\rho(G \cap U) < \frac{1}{k} \right\} \\ &= \overline{G} \cap \bigcap_{k=1}^{\infty} \bigcup_{U} U, \end{aligned}$$

where the union is taken over all open subsets U of X with $\operatorname{diam}_\rho f(G \cap U) < 1/k$. Moreover $\overline{G}$ is $\mathcal{G}_\delta$ in X since it is closed. It follows that A is $\mathcal{G}_\delta$ in X. Since A contains G, it suffices to prove that A is contained in G. Consider a point x in A and take a sequence $\{x_n\}$ in G converging to x. Since $\{x_n\}$ is a Cauchy sequence with respect to the metric ρ, the sequence $\{x_n\}$ converges to some point in G. Thus x is in G. ∎

Corollary 1.3 *The set $\mathbb{Q}$ is not $\mathcal{G}_\delta$ in $\mathbb{R}$.*

Proof. For every rational number r, the singleton $\{r\}$ is a closed subset of $\mathbb{Q}$ with empty interior. Since $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} \{r\}$, it follows from the Baire category theorem that $\mathbb{Q}$ is not completely metrizable. Therefore $\mathbb{Q}$ is not $\mathcal{G}_\delta$. ∎

2 The Set of Continuity Points of a Function

For a function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$, let $\operatorname{Cont} f$ denote the set of all points in $\mathbb{R}$ at which f is continuous.

Theorem 2.1 *Let A be a subset of $\mathbb{R}$. Then there exists a function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ with $\operatorname{Cont} f = A$ if and only if A is $\mathcal{G}_\delta$ in $\mathbb{R}$.*

Proof. We first prove the 'only if' part. Let $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ be a function. The oscillation $o(f, x)$ of f at a real number x is defined to be the infimum of $\operatorname{diam} f(U)$ over all open neighbourhoods U of x. Since

$$\operatorname{Cont} f = \{x \in \mathbb{R} \mid o(f,x) = 0\} = \bigcap_{n=1}^{\infty} \left\{ x \in \mathbb{R} \;\middle|\; o(f,x) < \frac{1}{n} \right\} = \bigcap_{n=1}^{\infty} \bigcup_{U} U,$$

where the union is taken over all open subsets U of $\mathbb{R}$ with $\operatorname{diam} f(U) < 1/n$, it follows that $\operatorname{Cont} f$ is $\mathcal{G}_\delta$ in $\mathbb{R}$.

Now we prove the 'if' part. Suppose that A is $\mathcal{G}_\delta$ in $\mathbb{R}$. We may take open subsets U_1, U_2, $\dots$ of $\mathbb{R}$ that satisfy $\mathbb{R} = U_1 \supset U_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} U_n = A$. Set $V_n = U_n \setminus U_{n+1}$

and $W_n = V_n \setminus (\operatorname{Int} V_n \cap \mathbb{Q})$. Since $V_1, V_2, \ldots$ are disjoint, so are $W_1, W_2, \ldots$. Therefore we may define a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} n^{-1} & \text{if } x \in W_n; \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove that $\operatorname{Cont} f = A$.

Assume that a is a real number in $A^c = \bigcup_{n=1}^{\infty} V_n$. Then there exists a unique positive integer n with $a \in V_n$. If $a \in \operatorname{Int} V_n \cap \mathbb{Q}$, then $f(a) = 0$ and each open neighbourhood of a has a point in W_n, at which the value of f is n^{-1}. If $a \in W_n$, then $f(a) = n^{-1}$ and each open neighbourhood of a has a point that does not belong to W_n, at which the value of f differs from n^{-1} by at least $1/n(n+1)$. In both cases, f is not continuous at a.

Assume that a is a real number in A. Since a belongs to no V_n, we have $f(a) = 0$. Let ε be a positive real number and take a positive integer n with $n^{-1} < \varepsilon$. Then U_n is an open neighbourhood of a such that every point x in U_n satisfies $0 \leqq f(x) \leqq n^{-1} < \varepsilon$. Thus f is continuous at a. ∎

Corollary 2.2 *There are no functions* $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ *for which* $\operatorname{Cont} f = \mathbb{Q}$.

Proof. This immediately follows from Corollary 1.3 and Theorem 2.1. ∎

Remark. A well-known function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ with $\operatorname{Cont} f = \mathbb{R} \setminus \mathbb{Q}$ is defined as follows: set $f(m/n) = 1/n$ for coprime integers m and n with $n > 0$, and $f(x) = 0$ for an irrational number x.

3 References

[1] Gelbaum B. R. and Olmsted J. M. H., *Counterexamples in Analysis*, Dover.

[2] Srivastava S. M., *A Course on Borel Sets*, Graduate Texts in Mathematics, 180, Springer-Verlag.