GENERATING FUNCTIONS FOR SUMS OF POLYNOMIAL MULTIPLE ZETA VALUES

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ABSTRACT. The sum formulas for multiple zeta(-star) values and symmetric multiple zeta(-star) values bear a striking resemblance. We explain the resemblance in a rather straightforward manner using an identity that involves the Schur multiple zeta values. We also obtain the sum formula for polynomial multiple zeta(-star) values in terms of generating functions, simultaneously generalizing the sum formulas for multiple zeta(-star) values and symmetric multiple zeta(-star) values.

CONTENTS

1. Introduction

1.1. Multiple zeta(-star) values and their sum formula.

An *index* is a finite sequence of positive integers, including the empty sequence $\emptyset$. If $\boldsymbol{k} = (k_1, \ldots, k_r)$ is an index, then we define its *weight* by $|\boldsymbol{k}| = k_1 + \cdots + k_r$ and its *depth* by $\mathrm{dep}\,\boldsymbol{k} = r$. We say that an index is *admissible* if either it is empty or its last component is greater than 1.

If $\boldsymbol{k} = (k_1, \ldots, k_r)$ is an admissible index, then we define the *multiple zeta value* and *multiple zeta-star value* by

$$\zeta(\boldsymbol{k}) = \sum_{1 \le m_1 < \cdots < m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}, \qquad \zeta^\star(\boldsymbol{k}) = \sum_{1 \le m_1 \le \cdots \le m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

respectively, where we understand that $\zeta(\emptyset) = \zeta^\star(\emptyset) = 1$. The multiple zeta(-star) values are known to satisfy a large number of relations, of which one of the most well-known is the *sum formula*. The sum formula asserts that the multiple zeta(-star) values of fixed weight and depth add up to an integer multiple of the Riemann zeta value:

Theorem 1.1 (sum formula for multiple zeta(-star) values; Granville [3], Zagier)**.** *If r is a nonnegative integer and w is an integer with $w \ge r + 2$, then*

$$\sum_{\substack{k_1 + \cdots + k_r + a = w \\ k_1, \ldots, k_r \ge 1 \\ a \ge 2}} \zeta(k_1, \ldots, k_r, a) = \zeta(w), \qquad \sum_{\substack{k_1 + \cdots + k_r + a = w \\ k_1, \ldots, k_r \ge 1 \\ a \ge 2}} \zeta^\star(k_1, \ldots, k_r, a) = \binom{w-1}{r} \zeta(w).$$

1.2. Regularization for multiple zeta(-star) values.

Let $\mathcal{Z}$ denote the $\mathbb{Q}$-linear space spanned by the multiple zeta values. As illustrated by

$$\zeta^\star(k_1, k_2) = \sum_{1 \le m_1 \le m_2} \frac{1}{m_1^{k_1} m_2^{k_2}} = \left(\sum_{1 \le m_1 < m_2} + \sum_{1 \le m_1 = m_2} \right) \frac{1}{m_1^{k_1} m_2^{k_2}}$$
$$= \zeta(k_1, k_2) + \zeta(k_1 + k_2),$$

the multiple zeta-star values are sums of multiple zeta values and therefore belong to $\mathcal{Z}$. Moreover, as illustrated by

$$\zeta(k)\zeta(l) = \left(\sum_{m=1}^\infty \frac{1}{m^k} \right) \left(\sum_{m=1}^\infty \frac{1}{m^l} \right) = \left(\sum_{1 \le m_1 < m_2} + \sum_{1 \le m_2 < m_1} + \sum_{1 \le m_1 = m_2} \right) \frac{1}{m_1^k m_2^l}$$
$$= \zeta(k, l) + \zeta(l, k) + \zeta(k + l),$$

the space $\mathcal{Z}$ is closed under multiplication, thereby being a $\mathbb{Q}$-algebra. Ihara, Kaneko, and Zagier [6] employed a method, called *regularization*, for defining the multiple zeta(-star) values for non-admissible indices as elements of the polynomial algebra $\mathcal{Z}[T]$, by assuming that those relations illustrated above hold even for non-admissible indices and setting $\zeta(1) = T$. For example, by

$$\zeta^\star(2, 1) = \zeta(2, 1) + \zeta(3), \qquad \zeta(2)\zeta(1) = \zeta(2, 1) + \zeta(1, 2) + \zeta(3),$$

we infer that $\zeta(2, 1) = \zeta(2)T - \zeta(1, 2) - \zeta(3)$ and $\zeta^\star(2, 1) = \zeta(2)T - \zeta(1, 2)$ in $\mathcal{Z}[T]$.

Remark 1.2. The reader familiar with regularization is reminded that this paper deals only with harmonic regularization, as opposed to shuffle regularization.

1.3. **Symmetric multiple zeta(-star) values and their sum formula.** If $\boldsymbol{k} = (k_1, \ldots, k_r)$ is an index, then we define

$$\zeta_S(\boldsymbol{k}) = \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta(k_1, \ldots, k_i) \zeta(k_r, \ldots, k_{i+1}),$$

$$\zeta_S^\star(\boldsymbol{k}) = \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta^\star(k_1, \ldots, k_i) \zeta^\star(k_r, \ldots, k_{i+1}).$$

Although $\zeta_S(\boldsymbol{k})$ and $\zeta_S^\star(\boldsymbol{k})$ a priori belong to $\mathcal{Z}[T]$, it turns out that they have constant terms only and so belong to $\mathcal{Z}$. Kaneko and Zagier [7] defined the *symmetric multiple zeta(-star) values* as $\zeta_S(\boldsymbol{k}), \zeta_S^\star(\boldsymbol{k}) \bmod \zeta(2)$ in $\mathcal{Z}/\zeta(2)\mathcal{Z}$, and the second author [8] established the sum formula for symmetric multiple zeta(-star) values:

Theorem 1.3 (sum formula for symmetric multiple zeta(-star) values; Murahara [8])**.** *If r and s are nonnegative integers and w is an integer with $w \geq r + s + 2$, then*

$$\sum_{\substack{k_1+\cdots+k_r+a+l_1+\cdots+l_s=w \\ k_1,\ldots,k_r,l_1,\ldots,l_s \geq 1 \\ a \geq 2}} \zeta_S(k_1, \ldots, k_r, a, l_1, \ldots, l_s) \equiv \left(-(-1)^r \binom{w-1}{r} + (-1)^s \binom{w-1}{s} \right) \zeta(w),$$

$$\sum_{\substack{k_1+\cdots+k_r+a+l_1+\cdots+l_s=w \\ k_1,\ldots,k_r,l_1,\ldots,l_s \geq 1 \\ a \geq 2}} \zeta_S^\star(k_1, \ldots, k_r, a, l_1, \ldots, l_s) \equiv \left((-1)^s \binom{w-1}{r} - (-1)^r \binom{w-1}{s} \right) \zeta(w)$$

modulo $\zeta(2)\mathcal{Z}$.

Remark 1.4. Note that our convention on the order of arguments is opposite to that of [8].

Although Theorems 1.1 and 1.3 bear a striking resemblance, no good reason has been offered thus far. We shall give an identity (Proposition 4.10) that together with a generalization of Theorem 1.1 implies Theorem 1.3, and so we have probably succeeded in explaining the resemblance to some extent.

1.4. **Restatement of the sum formulas in terms of generating functions.** We restate Theorems 1.1 and 1.3 in terms of generating functions. Define

$$\psi_1(W) = \sum_{k=2}^{\infty} \zeta(k) W^{k-1} \in \mathcal{Z}[[W]].$$

Remark 1.5. Our $\psi_1(W)$ is reminiscent of the digamma function, which satisfies

$$\psi(z+1) = -\gamma - \sum_{k=2}^{\infty} \zeta(k)(-z)^{k-1}.$$

Then Theorems 1.1 and 1.3 can be rephrased as follows (proofs will be given in Subsection 1.8):

Proposition 1.6. *We have*

$$\sum_{\substack{\boldsymbol{k} \\ a \geq 2}} \zeta(\boldsymbol{k}, a) A^{\operatorname{dep} \boldsymbol{k}} W^{|\boldsymbol{k}|+a} = \frac{W}{1-A}(\psi_1(W) - \psi_1(AW)),$$

$$\sum_{\substack{\boldsymbol{k} \\ a \geq 2}} \zeta^\star(\boldsymbol{k}, a) A^{\operatorname{dep} \boldsymbol{k}} W^{|\boldsymbol{k}|+a} = W(\psi_1((1+A)W) - \psi_1(AW))$$

in $\mathcal{Z}[A][[W]]$.

Proposition 1.7. *We have*

$$\sum_{\substack{\boldsymbol{k}, \boldsymbol{l} \\ a \geq 2}} \zeta_S(\boldsymbol{k}, a, \boldsymbol{l}) A^{\operatorname{dep} \boldsymbol{k}} B^{\operatorname{dep} \boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$\equiv -\frac{W}{1-B}(\psi_1((1-A)W) - \psi_1((B-A)W)) + \frac{W}{1-A}(\psi_1((1-B)W) - \psi_1((A-B)W)),$$

$$\sum_{\substack{\boldsymbol{k}, \boldsymbol{l} \\ a \geq 2}} \zeta_S^\star(\boldsymbol{k}, a, \boldsymbol{l}) A^{\operatorname{dep} \boldsymbol{k}} B^{\operatorname{dep} \boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$\equiv \frac{W}{1+B}(\psi_1((1+A)W) - \psi_1((A-B)W)) - \frac{W}{1+A}(\psi_1((1+B)W) - \psi_1((B-A)W))$$

modulo $\zeta(2)\mathcal{Z}$ *in* $\mathcal{Z}[A, B][[W]]$.

1.5. **Polynomial multiple zeta(-star) values.** If $\boldsymbol{k} = (k_1, \ldots, k_r)$ is an index, then the authors ([4]) defined the *polynomial multiple zeta(-star) value* by

$$\zeta_{x,y}(\boldsymbol{k}) = \sum_{i=0}^r \zeta(k_1, \ldots, k_i)\zeta(k_r, \ldots, k_{i+1}) x^{k_1+\cdots+k_i} y^{k_{i+1}+\cdots+k_r} \in \mathcal{Z}[T][x, y],$$

$$\zeta_{x,y}^\star(\boldsymbol{k}) = \sum_{i=0}^r \zeta^\star(k_1, \ldots, k_i)\zeta^\star(k_r, \ldots, k_{i+1}) x^{k_1+\cdots+k_i} y^{k_{i+1}+\cdots+k_r} \in \mathcal{Z}[T][x, y].$$

Notice that the polynomial multiple zeta(-star) values are a common generalization of $\zeta^{(\star)}(\boldsymbol{k})$ and $\zeta_S^{(\star)}(\boldsymbol{k})$:

$$\zeta_{1,0}(\boldsymbol{k}) = \zeta(\boldsymbol{k}), \quad \zeta_{1,0}^\star(\boldsymbol{k}) = \zeta^\star(\boldsymbol{k}), \quad \zeta_{1,-1}(\boldsymbol{k}) = \zeta_S(\boldsymbol{k}), \quad \zeta_{1,-1}^\star(\boldsymbol{k}) = \zeta_S^\star(\boldsymbol{k}).$$

1.6. **Main theorem.** Our main theorem computes the generating functions in Proposition 1.7 with ζ_S replaced by $\zeta_{x,y}$. To state the theorem, we need to define

$$\Gamma_1(W) = \exp\left(\sum_{k=1}^\infty \frac{\zeta(k)}{k} W^k\right) \in \mathcal{Z}[T][[W]].$$

Remark 1.8. Our $\Gamma_1(W)$ is reminiscent of the gamma function, which satisfies

$$\Gamma(z+1) = \exp\left(-\gamma z + \sum_{k=2}^\infty \frac{\zeta(k)}{k}(-z)^k\right).$$

Note also that $\exp(-TW)\Gamma_1(W) \in \mathcal{Z}[[W]]$, and that

$$A(W) = \exp(TW)\Gamma_1(-W) = \exp\left(\sum_{k=2}^\infty \frac{(-1)^k}{k}\zeta(k)W^k\right)$$

played an essential role in the regularization theorem due to Ihara, Kaneko, and Zagier [6].

Theorem 1.9 (Main theorem; Theorem 4.12). *We have*

$$
\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) A^{\mathrm{dep}\,\boldsymbol{k}} B^{\mathrm{dep}\,\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}
$$

$$
= \frac{yW}{1-B}(\psi_1(y(1-A)W) - \psi_1(y(B-A)W))\frac{\Gamma_1(xW)\Gamma_1(yW)}{\Gamma_1(x(1-A)W)\Gamma_1(y(1-A)W)}
$$

$$
+ \frac{xW}{1-A}(\psi_1(x(1-B)W) - \psi_1(x(A-B)W))\frac{\Gamma_1(xW)\Gamma_1(yW)}{\Gamma_1(x(1-B)W)\Gamma_1(y(1-B)W)},
$$

$$
\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} \zeta_{x,y}^{\star}(\boldsymbol{k},a,\boldsymbol{l}) A^{\mathrm{dep}\,\boldsymbol{k}} B^{\mathrm{dep}\,\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}
$$

$$
= \frac{yW}{1+A}(\psi_1(y(1+B)W) - \psi_1(y(B-A)W))\frac{\Gamma_1(x(1+A)W)\Gamma_1(y(1+A)W)}{\Gamma_1(xW)\Gamma_1(yW)}
$$

$$
+ \frac{xW}{1+B}(\psi_1(x(1+A)W) - \psi_1(x(A-B)W))\frac{\Gamma_1(x(1+B)W)\Gamma_1(y(1+B)W)}{\Gamma_1(xW)\Gamma_1(yW)}
$$

in $\mathcal{Z}[T][x,y][A,B][[W]]$.

1.7. Corollaries of our main theorem.

Corollary 1.10. *If r and s are nonnegative integers and w is an integer with $w \geq r + s + 2$, then*

$$
\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \mathrm{dep}\,\boldsymbol{k}=r, \mathrm{dep}\,\boldsymbol{l}=s \\ a \geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}), \quad \sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \mathrm{dep}\,\boldsymbol{k}=r, \mathrm{dep}\,\boldsymbol{l}=s \\ a \geq 2}} \zeta_{x,y}^{\star}(\boldsymbol{k},a,\boldsymbol{l}) \in \mathbb{Q}[T,\zeta(2),\ldots,\zeta(w)][x,y].
$$

Proof. The corollary follows from Theorem 1.9 and the observation that the coefficients of $1, W, \ldots, W^{w-1}$ in $\psi_1(W)$ and those of $1, W, \ldots, W^w$ in $\Gamma_1(W)$ and $\Gamma_1(W)^{-1}$ belong to $\mathbb{Q}[T,\zeta(2),\ldots,\zeta(w)][x,y]$. $\qquad\square$

Example 1.11. Consider the case $w = 4$. Then

$$
\sum_{\substack{r+s \leq 2 \\ r,s \geq 0}} \sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=r, \mathrm{dep}\,\boldsymbol{l}=s \\ a \geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) A^r B^s,
$$

being the coefficient of W^4 in

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) A^{\operatorname{dep}\boldsymbol{k}} B^{\operatorname{dep}\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= \frac{yW}{1-B}(\psi_1(y(1-A)W) - \psi_1(y(B-A)W))\frac{\Gamma_1(xW)\Gamma_1(yW)}{\Gamma_1(x(1-A)W)\Gamma_1(y(1-A)W)}$$

$$+ \frac{xW}{1-A}(\psi_1(x(1-B)W) - \psi_1(x(A-B)W))\frac{\Gamma_1(xW)\Gamma_1(yW)}{\Gamma_1(x(1-B)W)\Gamma_1(y(1-B)W)}$$

$$= \frac{1}{1-B}\left(\sum_{k=2}^{\infty}\zeta(k)((1-A)^{k-1} - (B-A)^{k-1})y^k W^k\right)\exp\left(\sum_{k=1}^{\infty}\frac{\zeta(k)}{k}(x^k+y^k)(1-(1-A)^k)W^k\right)$$

$$+ \frac{1}{1-A}\left(\sum_{k=2}^{\infty}\zeta(k)((1-B)^{k-1} - (A-B)^{k-1})x^k W^k\right)\exp\left(\sum_{k=1}^{\infty}\frac{\zeta(k)}{k}(x^k+y^k)(1-(1-B)^k)W^k\right),$$

is equal to

$$\zeta(4)(3A^2 - 3AB + B^2 - 3A + B + 1)y^4 + \zeta(3)T(-2A + B + 1)y^3(x+y)A$$

$$+ \zeta(2)y^2\left(\frac{\zeta(2)}{2}(x^2+y^2)(2A - A^2) + \frac{1}{2}T^2(x+y)^2 A^2\right)$$

$$+ \zeta(4)(A^2 - 3AB + 3B^2 + A - 3B + 1)x^4 + \zeta(3)T(A - 2B + 1)x^3(x+y)B$$

$$+ \zeta(2)x^2\left(\frac{\zeta(2)}{2}(x^2+y^2)(2B - B^2) + \frac{1}{2}T^2(x+y)^2 B^2\right).$$

We therefore have

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(x^4+y^4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=1,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(x^4 - 3y^4) + \zeta(3)Ty^3(x+y) + \zeta(2)^2 y^2(x^2+y^2),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=1 \\ a\geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(-3x^4 + y^4) + \zeta(3)Tx^3(x+y) + \zeta(2)^2 x^2(x^2+y^2),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=2,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(x^4 + 3y^4) - 2\zeta(3)Ty^3(x+y) - \frac{\zeta(2)^2}{2}y^2(x^2+y^2) + \frac{\zeta(2)}{2}T^2 y^2(x+y)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=1,\operatorname{dep}\boldsymbol{l}=1 \\ a\geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = -3\zeta(4)(x^4+y^4) + \zeta(3)T(x+y)(x^3+y^3),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=2 \\ a\geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(3x^4 + y^4) - 2\zeta(3)Tx^3(x+y) - \frac{\zeta(2)^2}{2}x^2(x^2+y^2) + \frac{\zeta(2)}{2}T^2 x^2(x+y)^2.$$

In a similar manner, we have

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=0, \mathrm{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta^\star_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(x^4+y^4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=1, \mathrm{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta^\star_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(3x^4-y^4) + \zeta(3)Ty^3(x+y) + \zeta(2)^2 y^2(x^2+y^2),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=0, \mathrm{dep}\,\boldsymbol{l}=1 \\ a\geq 2}} \zeta^\star_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(-x^4+3y^4) + \zeta(3)Tx^3(x+y) + \zeta(2)^2 x^2(x^2+y^2),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=2, \mathrm{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta^\star_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(3x^4+y^4) - \zeta(3)Ty^3(x+y) + \frac{\zeta(2)^2}{2}y^2(x^2+y^2) + \frac{\zeta(2)}{2}T^2y^2(x+y)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=1, \mathrm{dep}\,\boldsymbol{l}=1 \\ a\geq 2}} \zeta^\star_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = -3\zeta(4)(x^4+y^4) + 2\zeta(3)T(x+y)(x^3+y^3),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=0, \mathrm{dep}\,\boldsymbol{l}=2 \\ a\geq 2}} \zeta^\star_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) = \zeta(4)(x^4+3y^4) - \zeta(3)Tx^3(x+y) + \frac{\zeta(2)^2}{2}x^2(x^2+y^2) + \frac{\zeta(2)}{2}T^2x^2(x+y)^2.$$

Corollary 1.12. *We have*

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} \zeta(\boldsymbol{k},a,\boldsymbol{l})A^{\mathrm{dep}\,\boldsymbol{k}}B^{\mathrm{dep}\,\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} = \frac{W}{1-A}(\psi_1((1-B)W) - \psi_1((A-B)W))\frac{\Gamma_1(W)}{\Gamma_1((1-B)W)},$$

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} \zeta^\star(\boldsymbol{k},a,\boldsymbol{l})A^{\mathrm{dep}\,\boldsymbol{k}}B^{\mathrm{dep}\,\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} = \frac{W}{1+B}(\psi_1((1+A)W) - \psi_1((A-B)W))\frac{\Gamma_1((1+B)W)}{\Gamma_1(W)}$$

in $\mathcal{Z}[T][A,B][[W]]$.

Proof. Set $x=1$ and $y=0$ in Theorem 1.9, and observe that $\Gamma_1(0)=1$. $\square$

Remark 1.13. Corollary 1.12 is a generalization of Proposition 1.6 (or equivalently of Theorem 1.1); indeed, setting $B=0$ in Corollary 1.12 gives Proposition 1.6.

Corollary 1.14. *If r and s are nonnegative integers and w is an integer with $w \geq r+s+2$, then*

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \mathrm{dep}\,\boldsymbol{k}=r, \mathrm{dep}\,\boldsymbol{l}=s \\ a\geq 2}} \zeta(\boldsymbol{k},a,\boldsymbol{l}), \quad \sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \mathrm{dep}\,\boldsymbol{k}=r, \mathrm{dep}\,\boldsymbol{l}=s \\ a\geq 2}} \zeta^\star(\boldsymbol{k},a,\boldsymbol{l}) \in \mathbb{Q}[T,\zeta(2),\dots,\zeta(w)].$$

Proof. Immediate from Corollary 1.10 (or Corollary 1.12). $\square$

Example 1.15. Setting $x = 1$ and $y = 0$ in Example 1.11 gives

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=0,\text{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta(\boldsymbol{k}, a, \boldsymbol{l}) = \zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=1,\text{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta(\boldsymbol{k}, a, \boldsymbol{l}) = \zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=0,\text{dep}\,\boldsymbol{l}=1 \\ a\geq 2}} \zeta(\boldsymbol{k}, a, \boldsymbol{l}) = -3\zeta(4) + \zeta(3)T + \zeta(2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=2,\text{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta(\boldsymbol{k}, a, \boldsymbol{l}) = \zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=1,\text{dep}\,\boldsymbol{l}=1 \\ a\geq 2}} \zeta(\boldsymbol{k}, a, \boldsymbol{l}) = -3\zeta(4) + \zeta(3)T,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=0,\text{dep}\,\boldsymbol{l}=2 \\ a\geq 2}} \zeta(\boldsymbol{k}, a, \boldsymbol{l}) = 3\zeta(4) - 2\zeta(3)T - \frac{\zeta(2)^2}{2} + \frac{\zeta(2)}{2}T^2$$

and

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=0,\text{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta^\star(\boldsymbol{k}, a, \boldsymbol{l}) = \zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=1,\text{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta^\star(\boldsymbol{k}, a, \boldsymbol{l}) = 3\zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=0,\text{dep}\,\boldsymbol{l}=1 \\ a\geq 2}} \zeta^\star(\boldsymbol{k}, a, \boldsymbol{l}) = -\zeta(4) + \zeta(3)T + \zeta(2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=2,\text{dep}\,\boldsymbol{l}=0 \\ a\geq 2}} \zeta^\star(\boldsymbol{k}, a, \boldsymbol{l}) = 3\zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=1,\text{dep}\,\boldsymbol{l}=1 \\ a\geq 2}} \zeta^\star(\boldsymbol{k}, a, \boldsymbol{l}) = -3\zeta(4) + 2\zeta(3)T,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \text{dep}\,\boldsymbol{k}=0,\text{dep}\,\boldsymbol{l}=2 \\ a\geq 2}} \zeta^\star(\boldsymbol{k}, a, \boldsymbol{l}) = \zeta(4) - \zeta(3)T + \frac{\zeta(2)^2}{2} + \frac{\zeta(2)}{2}T^2.$$

Corollary 1.16. *We have*

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} \zeta_S(\boldsymbol{k}, a, \boldsymbol{l}) A^{\operatorname{dep} \boldsymbol{k}} B^{\operatorname{dep} \boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= -\frac{W}{1-B}(\psi_1(-(1-A)W) - \psi_1((A-B)W))\frac{\pi W}{\sin \pi W} \cdot \frac{\sin \pi (1-A)W}{\pi(1-A)W}$$

$$+ \frac{W}{1-A}(\psi_1((1-B)W) - \psi_1((A-B)W))\frac{\pi W}{\sin \pi W} \cdot \frac{\sin \pi (1-B)W}{\pi(1-B)W},$$

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} \zeta_S^{\star}(\boldsymbol{k}, a, \boldsymbol{l}) A^{\operatorname{dep} \boldsymbol{k}} B^{\operatorname{dep} \boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= -\frac{W}{1+A}(\psi_1(-(1+B)W) - \psi_1((A-B)W))\frac{\sin \pi W}{\pi W} \cdot \frac{\pi(1+A)W}{\sin \pi (1+A)W}$$

$$+ \frac{W}{1+B}(\psi_1((1+A)W) - \psi_1((A-B)W))\frac{\sin \pi W}{\pi W} \cdot \frac{\pi(1+B)W}{\sin \pi (1+B)W}$$

in $\mathcal{Z}[A,B][[W]]$.

Proof. Set $x = 1$ and $y = -1$ in Theorem 1.9, and use the identity $\Gamma_1(W)\Gamma_1(-W) = \pi W / \sin \pi W$, whose proof will be given as Lemma 1.19 in Subsection 1.8. $\square$

Corollary 1.17. *If r and s are nonnegative integers and w is an integer with $w \geq r + s + 2$, then*

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \operatorname{dep} \boldsymbol{k}=r, \operatorname{dep} \boldsymbol{l}=s \\ a \geq 2}} \zeta_S(\boldsymbol{k}, a, \boldsymbol{l}), \quad \sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \operatorname{dep} \boldsymbol{k}=r, \operatorname{dep} \boldsymbol{l}=s \\ a \geq 2}} \zeta_S^{\star}(\boldsymbol{k}, a, \boldsymbol{l}) \in \mathbb{Q}[\zeta(2), \ldots, \zeta(w)].$$

Proof. Immediate from Corollary 1.16. $\square$

Example 1.18. Setting $x = 1$ and $y = -1$ in Example 1.11 gives

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_S(\boldsymbol{k},a,\boldsymbol{l}) = 2\zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=1,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_S(\boldsymbol{k},a,\boldsymbol{l}) = -2\zeta(4) + 2\zeta(2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=1 \\ a\geq 2}} \zeta_S(\boldsymbol{k},a,\boldsymbol{l}) = -2\zeta(4) + 2\zeta(2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=2,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_S(\boldsymbol{k},a,\boldsymbol{l}) = 4\zeta(4) - \zeta(2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=1,\operatorname{dep}\boldsymbol{l}=1 \\ a\geq 2}} \zeta_S(\boldsymbol{k},a,\boldsymbol{l}) = -6\zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=2 \\ a\geq 2}} \zeta_S(\boldsymbol{k},a,\boldsymbol{l}) = 4\zeta(4) - \zeta(2)^2$$

and

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_S^{\star}(\boldsymbol{k},a,\boldsymbol{l}) = 2\zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=1,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_S^{\star}(\boldsymbol{k},a,\boldsymbol{l}) = 2\zeta(4) + 2\zeta(2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=1 \\ a\geq 2}} \zeta_S^{\star}(\boldsymbol{k},a,\boldsymbol{l}) = 2\zeta(4) + 2\zeta(2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=2,\operatorname{dep}\boldsymbol{l}=0 \\ a\geq 2}} \zeta_S^{\star}(\boldsymbol{k},a,\boldsymbol{l}) = 4\zeta(4) + \zeta(2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=1,\operatorname{dep}\boldsymbol{l}=1 \\ a\geq 2}} \zeta_S^{\star}(\boldsymbol{k},a,\boldsymbol{l}) = -6\zeta(4),$$

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=4 \\ \operatorname{dep}\boldsymbol{k}=0,\operatorname{dep}\boldsymbol{l}=2 \\ a\geq 2}} \zeta_S^{\star}(\boldsymbol{k},a,\boldsymbol{l}) = 4\zeta(4) + \zeta(2)^2.$$

1.8. **Proofs of propositions and an identity stated in this section.**

Proof of Proposition 1.6. Theorem 1.1 shows that

$$\sum_{\substack{\boldsymbol{k} \\ a \geq 2}} \zeta(\boldsymbol{k}, a) A^{\operatorname{dep} \boldsymbol{k}} W^{|\boldsymbol{k}|+a} = \sum_{\substack{r \geq 0 \\ w \geq r+2}} \sum_{\substack{|\boldsymbol{k}|+a=w \\ \operatorname{dep} \boldsymbol{k}=r \\ a \geq 2}} \zeta(\boldsymbol{k}, a) A^r W^w = \sum_{\substack{r \geq 0 \\ w \geq r+2}} \zeta(w) A^r W^w$$

$$= \sum_{w=2}^{\infty} \sum_{r=0}^{w-2} A^r \zeta(w) W^w = \sum_{w=2}^{\infty} \frac{1 - A^{w-1}}{1 - A} \zeta(w) W^w$$

$$= \frac{W}{1 - A} (\psi_1(W) - \psi_1(AW))$$

and

$$\sum_{\substack{\boldsymbol{k} \\ a \geq 2}} \zeta^{\star}(\boldsymbol{k}, a) A^{\operatorname{dep} \boldsymbol{k}} W^{|\boldsymbol{k}|+a} = \sum_{\substack{r \geq 0 \\ w \geq r+2}} \sum_{\substack{|\boldsymbol{k}|+a=w \\ \operatorname{dep} \boldsymbol{k}=r \\ a \geq 2}} \zeta^{\star}(\boldsymbol{k}, a) A^r W^w$$

$$= \sum_{\substack{r \geq 0 \\ w \geq r+2}} \binom{w-1}{r} \zeta(w) A^r W^w$$

$$= \sum_{w=2}^{\infty} \sum_{r=0}^{w-2} \binom{w-1}{r} A^r \zeta(w) W^w$$

$$= \sum_{w=2}^{\infty} ((1+A)^{w-1} - A^{w-1}) \zeta(w) W^w$$

$$= W(\psi_1((1+A)W) - \psi_1(AW)),$$

as required. $\qquad\qquad\square$

Proof of Proposition 1.7. Theorem 1.3 shows that

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} \zeta_S(\boldsymbol{k}, a, \boldsymbol{l}) A^{\operatorname{dep} \boldsymbol{k}} B^{\operatorname{dep} \boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= \sum_{\substack{r,s \geq 0 \\ w \geq r+s+2}} \sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \operatorname{dep} \boldsymbol{k}=r, \operatorname{dep} \boldsymbol{l}=s \\ a \geq 2}} \zeta_S(\boldsymbol{k}, a, \boldsymbol{l}) A^r B^s W^w$$

$$\equiv \sum_{\substack{r,s \geq 0 \\ w \geq r+s+2}} \left(-(-1)^r \binom{w-1}{r} + (-1)^s \binom{w-1}{s} \right) \zeta(w) A^r B^s W^w$$

$$= \sum_{w=2}^{\infty} \left(-\sum_{r=0}^{w-2} \sum_{s=0}^{w-r-2} \binom{w-1}{r} (-A)^r B^s + \sum_{s=0}^{w-2} \sum_{r=0}^{w-s-2} \binom{w-1}{s} A^r (-B)^s \right) \zeta(w) W^w$$

$$= \sum_{w=2}^{\infty} \left(-\sum_{r=0}^{w-2} \binom{w-1}{r} (-A)^r \frac{1 - B^{w-r-1}}{1 - B} + \sum_{s=0}^{w-2} \binom{w-1}{s} \frac{1 - A^{w-s-1}}{1 - A} (-B)^s \right) \zeta(w) W^w$$

$$= \sum_{w=2}^{\infty} \left(-\frac{(1-A)^{w-1} - (B-A)^{w-1}}{1 - B} + \frac{(1-B)^{w-1} - (A-B)^{w-1}}{1 - A} \right) \zeta(w) W^w$$

$$= -\frac{W}{1 - B}(\psi_1((1-A)W) - \psi_1((B-A)W)) + \frac{W}{1 - A}(\psi_1((1-B)W) - \psi_1((A-B)W))$$

and

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} \zeta_S^\star(\boldsymbol{k},a,\boldsymbol{l}) A^{\operatorname{dep}\boldsymbol{k}} B^{\operatorname{dep}\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= \sum_{\substack{r,s \geq 0 \\ w \geq r+s+2}} \sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \operatorname{dep}\boldsymbol{k}=r,\operatorname{dep}\boldsymbol{l}=s \\ a \geq 2}} \zeta_S^\star(\boldsymbol{k},a,\boldsymbol{l}) A^r B^s W^w$$

$$\equiv \sum_{\substack{r,s \geq 0 \\ w \geq r+s+2}} \left((-1)^s \binom{w-1}{r} - (-1)^r \binom{w-1}{s} \right) \zeta(w) A^r B^s W^w$$

$$= \sum_{w=2}^{\infty} \left(\sum_{r=0}^{w-2} \sum_{s=0}^{w-r-2} \binom{w-1}{r} A^r (-B)^s - \sum_{s=0}^{w-2} \sum_{r=0}^{w-s-2} \binom{w-1}{s} (-A)^r B^s \right) \zeta(w) W^w$$

$$= \sum_{w=2}^{\infty} \left(\sum_{r=0}^{w-2} \binom{w-1}{r} A^r \frac{1-(-B)^{w-r-1}}{1+B} - \sum_{s=0}^{w-2} \binom{w-1}{s} \frac{1-(-A)^{w-s-1}}{1+A} B^s \right) \zeta(w) W^w$$

$$= \sum_{w=2}^{\infty} \left(\frac{(1+A)^{w-1} - (A-B)^{w-1}}{1+B} - \frac{(1+B)^{w-1} - (B-A)^{w-1}}{1+A} \right) \zeta(w) W^w$$

$$= \frac{W}{1+B} (\psi_1((1+A)W) - \psi_1((A-B)W)) - \frac{W}{1+A} (\psi_1((1+B)W) - \psi_1((B-A)W)),$$

as required. □

Lemma 1.19. *We have*

$$\Gamma_1(W)\Gamma_1(-W) = \frac{\pi W}{\sin \pi W}$$

in $\mathcal{Z}[[W]]$.

Proof. Since

$$\log(\Gamma_1(W)\Gamma_1(-W)) = \sum_{k=1}^{\infty} \frac{\zeta(k)}{k}(W^k + (-W)^k) = \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} W^{2k}$$

and

$$\log \frac{\pi W}{\sin \pi W} = \log \prod_{m=1}^{\infty} \left(1 - \frac{W^2}{m^2} \right)^{-1} = -\sum_{m=1}^{\infty} \log \left(1 - \frac{W^2}{m^2} \right) = \sum_{k,m=1}^{\infty} \frac{W^{2k}}{km^{2k}} = \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} W^{2k},$$

the lemma follows. □

2. Hopf algebra formed by the indices

We first recall Hoffman's result ([5]) that the indices form a Hopf algebra. We associate to each index $\boldsymbol{k} = (k_1,\ldots,k_r)$ a formal symbol $[\boldsymbol{k}] = [k_1,\ldots,k_r]$, and write $\mathcal{I}$ for the $\mathbb{Q}$-linear space of all formal $\mathbb{Q}$-linear combinations of the symbols $[\boldsymbol{k}]$ (introducing such formal symbols facilitates distinction, for example, between $2(k+l) \in \mathbb{Z}$ and $2[k+l] \in \mathcal{I}$).

For ease of notation, if $\boldsymbol{k} = (k_1,\ldots,k_r)$ is an index, then we write $\boldsymbol{k}_i = (k_1,\ldots,k_i)$ and $\boldsymbol{k}^i = (k_{i+1},\ldots,k_r)$ for $i = 0,\ldots,r$, where we understand that $\boldsymbol{k}_0 = \boldsymbol{k}^r = \emptyset$, and we write $\overleftarrow{\boldsymbol{k}} = (k_r,\ldots,k_1)$.

We now define the linear maps that make $\mathcal{I}$ a Hopf algebra. The multiplication $\mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$, often written as a bilinear product $*$ on $\mathcal{I}$ (known as the *harmonic product* or the *stuffle product*), is defined inductively by setting

(1) $[\boldsymbol{k}] * [\emptyset] = [\emptyset] * [\boldsymbol{k}] = [\boldsymbol{k}]$ whenever $\boldsymbol{k}$ is an index, and

(2) $[\boldsymbol{k}, k] * [\boldsymbol{l}, l] = [[\boldsymbol{k}, k] * [\boldsymbol{l}], l] + [[\boldsymbol{k}] * [\boldsymbol{l}, l], k] + [[\boldsymbol{k}] * [\boldsymbol{l}], k + l]$ whenever $\boldsymbol{k}$ and $\boldsymbol{l}$ are indices and k and l are positive integers, where on the right-hand side we understand that $[\cdot, l]$, $[\cdot, k]$, and $[\cdot, k + l]$ denote the $\mathbb{Q}$-linear operators of concatenating the specified integers.

The unit $\mathbb{Q} \to \mathcal{I}$ is given by $1 \mapsto [\emptyset]$. The comultiplication $\mathcal{I} \to \mathcal{I} \otimes \mathcal{I}$ is defined by

$$[\boldsymbol{k}] \mapsto \sum_{i=0}^{r} [\boldsymbol{k}_i] \otimes [\boldsymbol{k}^i]$$

for indices $\boldsymbol{k}$ of depth r. The counit $\mathcal{I} \to \mathbb{Q}$ is given by

$$[\boldsymbol{k}] \mapsto \begin{cases} 1 & \text{if } \boldsymbol{k} = \emptyset; \\ 0 & \text{otherwise} \end{cases}$$

for indices $\boldsymbol{k}$. The antipode $S \colon \mathcal{I} \to \mathcal{I}$ is given by

$$S([\boldsymbol{k}]) = (-1)^r [\overleftarrow{\boldsymbol{k}}]^\star,$$

for indices $\boldsymbol{k}$ of depth r. Here if $\boldsymbol{l} = (l_1, \ldots, l_s)$ is an index, then $[\boldsymbol{l}]^\star$ denotes the sum of all $[l_1 \square \cdots \square l_s]$ with each square replaced by a plus sign or a comma.

Theorem 2.1 (Hoffman [5]). *The maps given above make $\mathcal{I}$ a commutative Hopf algebra.*

In particular we have the following:

- The comultiplication $\mathcal{I} \to \mathcal{I} \otimes \mathcal{I}$ is an algebra homomorphism.
- The antipode $S \colon \mathcal{I} \to \mathcal{I}$ is an involution and algebra homomorphism. In this paper we find it more convenient to use the $\mathbb{Q}$-linear map $\tilde{S} \colon \mathcal{I} \to \mathcal{I}$ defined by $\tilde{S}([\boldsymbol{k}]) = (-1)^r [\boldsymbol{k}]^\star$ for indices $\boldsymbol{k}$ of depth r; it follows that $\tilde{S}$ is also an involution and algebra homomorphism.
- If $\boldsymbol{k}$ is an index of depth r, then

$$\sum_{i=0}^{r} (-1)^{r-i} [\boldsymbol{k}_i] * [\overleftarrow{\boldsymbol{k}^i}]^\star = \begin{cases} [\emptyset] & \text{if } \boldsymbol{k} = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

3. Generating functions for symmetric sums

3.1. Generating functions of $[\boldsymbol{k}]$ and $[\boldsymbol{k}]^\star$.
In this subsection, we compute the generating functions

$$\sum_{\boldsymbol{k}} [\boldsymbol{k}] A^{\operatorname{dep} \boldsymbol{k}} W^{|\boldsymbol{k}|}, \qquad \sum_{\boldsymbol{k}} [\boldsymbol{k}]^\star A^{\operatorname{dep} \boldsymbol{k}} W^{|\boldsymbol{k}|}$$

in $\mathcal{I}[A][[W]]$. To state the results, it is convenient to define the formal power series

$$\Gamma_{1,\mathcal{I}}(W) = \exp\left(\sum_{k=1}^{\infty} \frac{[k]}{k} W^k \right) \in \mathcal{I}[[W]].$$

Observe that

$$\tilde{S}(\Gamma_{1,\mathcal{I}}(W)) = \exp\left(-\sum_{k=1}^{\infty} \frac{[k]}{k} W^k \right) = \Gamma_{1,\mathcal{I}}(W)^{-1}.$$

Proposition 3.1. *We have*

$$\sum_{\boldsymbol{k}}[\boldsymbol{k}]A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|} = \frac{\Gamma_{1,\mathcal{I}}(W)}{\Gamma_{1,\mathcal{I}}((1-A)W)}, \qquad \sum_{\boldsymbol{k}}[\boldsymbol{k}]^{\star}A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|} = \frac{\Gamma_{1,\mathcal{I}}((1+A)W)}{\Gamma_{1,\mathcal{I}}(W)}$$

in $\mathcal{I}[A][[W]]$.

Proof. The first identity implies the second because

$$\sum_{\boldsymbol{k}}[\boldsymbol{k}]^{\star}A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|} = \tilde{S}\left(\sum_{\boldsymbol{k}}[\boldsymbol{k}](-A)^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|}\right) = \tilde{S}\left(\frac{\Gamma_{1,\mathcal{I}}(W)}{\Gamma_{1,\mathcal{I}}((1+A)W)}\right)$$

$$= \frac{\Gamma_{1,\mathcal{I}}((1+A)W)}{\Gamma_{1,\mathcal{I}}(W)}.$$

The first identity is equivalent to

$$\log\left(\sum_{\boldsymbol{k}}[\boldsymbol{k}]A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|}\right) = \sum_{k=1}^{\infty}\frac{[k]}{k}(1-(1-A)^{k})W^{k},$$

and since both sides have constant term 0 (with respect to W), it suffices to prove that both sides have the same derivative (with respect to W):

$$\frac{\sum_{\boldsymbol{k}\neq\emptyset}|\boldsymbol{k}|[\boldsymbol{k}]A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|-1}}{\sum_{\boldsymbol{k}}[\boldsymbol{k}]A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|}} = \sum_{k=1}^{\infty}[k](1-(1-A)^{k})W^{k-1},$$

which in turn is equivalent to

$$\left(\sum_{\boldsymbol{k}}[\boldsymbol{k}]A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|}\right)\left(\sum_{k=1}^{\infty}[k](1-(1-A)^{k})W^{k}\right) = \sum_{\boldsymbol{k}\neq\emptyset}|\boldsymbol{k}|[\boldsymbol{k}]A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|}.$$

For each nonempty index $\boldsymbol{l} = (l_1,\ldots,l_s)$, the coefficient of $[\boldsymbol{l}]W^{|\boldsymbol{l}|}$ in the left-hand side is

$$\sum_{j=1}^{s}A^{s-1}(1-(1-A)^{l_j}) + \sum_{j=1}^{s}A^{s}\sum_{i=1}^{l_j-1}(1-(1-A)^{i}),$$

which simplifies to $A^{s}\sum_{j=1}^{s}l_j = |\boldsymbol{l}|A^{\operatorname{dep}\boldsymbol{l}}$. $\qquad\qquad\square$

Remark 3.2. Substituting $A = 1$ and $A = -1$ into the equations in Proposition 3.1 respectively gives

$$\Gamma_{1,\mathcal{I}}(W) = \sum_{\boldsymbol{k}}[\boldsymbol{k}]W^{|\boldsymbol{k}|} = \sum_{k=0}^{\infty}[\{1\}^{k}]^{\star}W^{k},$$

$$\Gamma_{1,\mathcal{I}}(W)^{-1} = \sum_{\boldsymbol{k}}(-1)^{\operatorname{dep}\boldsymbol{k}}[\boldsymbol{k}]^{\star}W^{|\boldsymbol{k}|} = \sum_{k=0}^{\infty}(-1)^{k}[\{1\}^{k}]W^{k},$$

where $\{1\}^{k}$ denotes the index $(\underbrace{1,\ldots,1}_{k})$, which means $\emptyset$ if $k = 0$.

3.2. **Generating functions for symmetric sums of $[k]_{x,y}$ and $[k]^\star_{x,y}$.** If k is an index, then we define

$$[k]_{x,y} = \sum_{i=0}^{r} [k_i] * [\overleftarrow{k^i}] x^{|k_i|} y^{|k^i|} \in \mathcal{I}[x,y],$$

$$[k]^\star_{x,y} = \sum_{i=0}^{r} [k_i]^\star * [\overleftarrow{k^i}]^\star x^{|k_i|} y^{|k^i|} \in \mathcal{I}[x,y],$$

where $r = \operatorname{dep} k$. Note that if k is an index of depth r, then

$$\tilde{S}([k]_{x,y}) = \sum_{i=0}^{r} \tilde{S}([k_i]) * \tilde{S}([\overleftarrow{k^i}]) x^{|k_i|} y^{|k^i|}$$

$$= \sum_{i=0}^{r} (-1)^i [k_i]^\star * (-1)^{r-i} [\overleftarrow{k^i}]^\star x^{|k_i|} y^{|k^i|}$$

$$= (-1)^r [k]^\star_{x,y}.$$

Lemma 3.3. *The $\mathbb{Q}$-linear map from $\mathcal{I}$ to $\mathcal{I}[x,y]$ given by $[k] \mapsto [k]_{x,y}$ for indices k is an algebra homomorphism.*

Proof. The map in question is the composite

$$\mathcal{I} \to \mathcal{I} \otimes \mathcal{I}$$
$$\to \mathcal{I}[x] \otimes \mathcal{I}[y] \cong \mathcal{I} \otimes \mathbb{Q}[x] \otimes \mathcal{I} \otimes \mathbb{Q}[y] \cong \mathcal{I} \otimes \mathcal{I} \otimes \mathbb{Q}[x,y]$$
$$\to \mathcal{I} \otimes \mathbb{Q}[x,y] \cong \mathcal{I}[x,y],$$

where the arrows denote the comultiplication, the map $[k] \otimes [l] \mapsto [k] x^{|k|} \otimes [\overleftarrow{l}] y^{|l|}$, and the multiplication. $\square$

Proposition 3.4. *We have*

$$\sum_{k} [k]_{x,y} A^{\operatorname{dep} k} W^{|k|} = \frac{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}{\Gamma_{1,\mathcal{I}}(x(1-A)W)\Gamma_{1,\mathcal{I}}(y(1-A)W)},$$

$$\sum_{k} [k]^\star_{x,y} A^{\operatorname{dep} k} W^{|k|} = \frac{\Gamma_{1,\mathcal{I}}(x(1+A)W)\Gamma_{1,\mathcal{I}}(y(1+A)W)}{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}$$

in $\mathcal{I}[x,y][A][[W]]$.

Proof. The first identity implies the second because

$$\sum_{k} [k]^\star_{x,y} A^{\operatorname{dep} k} W^{|k|} = \tilde{S}\left(\sum_{k} [k]_{x,y} (-A)^{\operatorname{dep} k} W^{|k|}\right)$$

$$= \tilde{S}\left(\frac{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}{\Gamma_{1,\mathcal{I}}(x(1+A)W)\Gamma_{1,\mathcal{I}}(y(1+A)W)}\right)$$

$$= \frac{\Gamma_{1,\mathcal{I}}(x(1+A)W)\Gamma_{1,\mathcal{I}}(y(1+A)W)}{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}.$$

Since the algebra homomorphism $[k] \mapsto [k]_{x,y}$ satisfies

$$\Gamma_{1,\mathcal{I}}(W) \mapsto \exp\left(\sum_{k=1}^{\infty} \frac{[k]_{x,y}}{k} W^k\right) = \exp\left(\sum_{k=1}^{\infty} \frac{[k](x^k + y^k)}{k} W^k\right) = \Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW),$$

the first identity follows from Proposition 3.1. $\square$

3.3. Generating functions of $\zeta^{(\star)}(\boldsymbol{k})$, $\zeta_{x,y}^{(\star)}(\boldsymbol{k})$, and $\zeta_S^{(\star)}(\boldsymbol{k})$.
We define $\mathbb{Q}$-linear maps $Z\colon \mathcal{I} \to \mathcal{Z}[T]$, $Z_S\colon \mathcal{I} \to \mathcal{Z}$, and $Z_{x,y}\colon \mathcal{I} \to \mathcal{Z}[T][x,y]$ by setting $Z([\boldsymbol{k}]) = \zeta(\boldsymbol{k})$, $Z_S([\boldsymbol{k}]) = \zeta_S(\boldsymbol{k}) = Z([\boldsymbol{k}]_{1,-1})$, and $Z_{x,y}([\boldsymbol{k}]) = \zeta_{x,y}(\boldsymbol{k}) = Z([\boldsymbol{k}]_{x,y})$. Then they are all algebra homomorphisms, and satisfy $Z([\boldsymbol{k}]^\star) = \zeta^\star(\boldsymbol{k})$, $Z_S([\boldsymbol{k}]^\star) = \zeta_S^\star(\boldsymbol{k})$, and $Z_{x,y}([\boldsymbol{k}]^\star) = \zeta_{x,y}^\star(\boldsymbol{k})$.

We have $Z(\Gamma_{1,\mathcal{I}}(W)) = \Gamma_1(W)$, and Remark 3.2 shows that

$$\Gamma_1(W) = \exp\left(\sum_{k=1}^\infty \frac{\zeta(k)}{k}W^k\right) = \sum_{\boldsymbol{k}} \zeta(\boldsymbol{k})W^{|\boldsymbol{k}|} = \sum_{k=0}^\infty \zeta^\star(\{1\}^k)W^k,$$

$$\Gamma_1(W)^{-1} = \exp\left(-\sum_{k=1}^\infty \frac{\zeta(k)}{k}W^k\right) = \sum_{\boldsymbol{k}} (-1)^{\operatorname{dep}\boldsymbol{k}}\zeta^\star(\boldsymbol{k})W^{|\boldsymbol{k}|} = \sum_{k=0}^\infty (-1)^k\zeta(\{1\}^k)W^k.$$

Proposition 3.5. *We have*

$$\sum_{\boldsymbol{k}} \zeta(\boldsymbol{k})A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|} = \frac{\Gamma_1(W)}{\Gamma_1((1-A)W)}, \qquad \sum_{\boldsymbol{k}} \zeta^\star(\boldsymbol{k})A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|} = \frac{\Gamma_1((1+A)W)}{\Gamma_1(W)}$$

in $\mathcal{Z}[T][A][[W]]$.

Proof. Immediate from Proposition 3.1. $\qquad\square$

Proposition 3.6. *We have*

$$\sum_{\boldsymbol{k}} \zeta_{x,y}(\boldsymbol{k})A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|} = \frac{\Gamma_1(xW)\Gamma_1(yW)}{\Gamma_1(x(1-A)W)\Gamma_1(y(1-A)W)},$$

$$\sum_{\boldsymbol{k}} \zeta_{x,y}^\star(\boldsymbol{k})A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|} = \frac{\Gamma_1(x(1+A)W)\Gamma_1(y(1+A)W)}{\Gamma_1(xW)\Gamma_1(yW)}$$

in $\mathcal{Z}[T][x,y][A][[W]]$.

Proof. Immediate from Proposition 3.4. $\qquad\square$

Corollary 3.7. *If r is a nonnegative integer and w is an integer with $w \geq r$, then*

$$\sum_{\substack{|\boldsymbol{k}|=w \\ \operatorname{dep}\boldsymbol{k}=r}} \zeta_{x,y}(\boldsymbol{k}), \quad \sum_{\substack{|\boldsymbol{k}|=w \\ \operatorname{dep}\boldsymbol{k}=r}} \zeta_{x,y}^\star(\boldsymbol{k}) \in \mathbb{Q}[T,\zeta(2),\ldots,\zeta(w)][x,y].$$

Proof. Immediate from Proposition 3.6. $\qquad\square$

Example 3.8. Consider the case $w = 4$. Then

$$\sum_{r=0}^4 \sum_{\substack{|\boldsymbol{k}|=4 \\ \operatorname{dep}\boldsymbol{k}=r}} \zeta_{x,y}(\boldsymbol{k})A^r,$$

being the coefficient of W^4 in

$$\sum_{\boldsymbol{k}} \zeta_{x,y}(\boldsymbol{k})A^{\operatorname{dep}\boldsymbol{k}}W^{|\boldsymbol{k}|} = \frac{\Gamma_1(xW)\Gamma_1(yW)}{\Gamma_1(x(1-A)W)\Gamma_1(y(1-A)W)}$$

$$= \exp\left(\sum_{k=1}^\infty \frac{\zeta(k)}{k}(x^k + y^k)(1 - (1-A)^k)W^k\right),$$

is equal to

$$\frac{\zeta(4)}{4}(x^4 + y^4)(4A - 6A^2 + 4A^3 - A^4)$$

$$+ \frac{1}{2}\left(2 \cdot T(x+y)A \cdot \frac{\zeta(3)}{3}(x^3 + y^3)(3A - 3A^2 + A^3) + \left(\frac{\zeta(2)}{2}(x^2 + y^2)(2A - A^2)\right)^2\right)$$

$$+ \frac{1}{6} \cdot 3 \cdot (T(x+y)A)^2 \cdot \left(\frac{\zeta(2)}{2}(x^2 + y^2)(2A - A^2)\right)$$

$$+ \frac{1}{24}(T(x+y)A)^4.$$

We therefore have

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \text{dep } \boldsymbol{k}=0}} \zeta_{x,y}(\boldsymbol{k}) = 0,$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \text{dep } \boldsymbol{k}=1}} \zeta_{x,y}(\boldsymbol{k}) = \zeta(4)(x^4 + y^4),$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \text{dep } \boldsymbol{k}=2}} \zeta_{x,y}(\boldsymbol{k}) = -\frac{3}{2}\zeta(4)(x^4 + y^4) + \zeta(3)T(x+y)(x^3 + y^3) + \frac{\zeta(2)^2}{2}(x^2 + y^2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \text{dep } \boldsymbol{k}=3}} \zeta_{x,y}(\boldsymbol{k}) = \zeta(4)(x^4 + y^4) - \zeta(3)T(x+y)(x^3 + y^3) - \frac{\zeta(2)^2}{2}(x^2 + y^2)^2 + \frac{\zeta(2)}{2}T^2(x+y)^2(x^2 + y^2),$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \text{dep } \boldsymbol{k}=4}} \zeta_{x,y}(\boldsymbol{k}) = -\frac{\zeta(4)}{4}(x^4 + y^4) + \frac{\zeta(3)}{3}T(x+y)(x^3 + y^3) + \frac{\zeta(2)^2}{8}(x^2 + y^2)^2$$

$$- \frac{\zeta(2)}{4}T^2(x+y)^2(x^2 + y^2) + \frac{1}{24}T^4(x+y)^4.$$

In a similar manner, we have

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \operatorname{dep}\boldsymbol{k}=0}} \zeta^{\star}_{x,y}(\boldsymbol{k}) = 0,$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \operatorname{dep}\boldsymbol{k}=1}} \zeta^{\star}_{x,y}(\boldsymbol{k}) = \zeta(4)(x^4 + y^4),$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \operatorname{dep}\boldsymbol{k}=2}} \zeta^{\star}_{x,y}(\boldsymbol{k}) = \frac{3}{2}\zeta(4)(x^4 + y^4) + \zeta(3)T(x+y)(x^3+y^3) + \frac{\zeta(2)^2}{2}(x^2+y^2)^2,$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \operatorname{dep}\boldsymbol{k}=3}} \zeta^{\star}_{x,y}(\boldsymbol{k}) = \zeta(4)(x^4+y^4) + \zeta(3)T(x+y)(x^3+y^3) + \frac{\zeta(2)^2}{2}(x^2+y^2)^2 + \frac{\zeta(2)}{2}T^2(x+y)^2(x^2+y^2),$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \operatorname{dep}\boldsymbol{k}=4}} \zeta^{\star}_{x,y}(\boldsymbol{k}) = \frac{\zeta(4)}{4}(x^4+y^4) + \frac{\zeta(3)}{3}T(x+y)(x^3+y^3) + \frac{\zeta(2)^2}{8}(x^2+y^2)^2$$

$$+ \frac{\zeta(2)}{4}T^2(x+y)^2(x^2+y^2) + \frac{1}{24}T^4(x+y)^4.$$

Proposition 3.9. *We have*

$$\sum_{\boldsymbol{k}} \zeta_S(\boldsymbol{k}) A^{\operatorname{dep}\boldsymbol{k}} W^{|\boldsymbol{k}|} = \frac{\pi W}{\sin \pi W} \cdot \frac{\sin \pi(1-A)W}{\pi(1-A)W},$$

$$\sum_{\boldsymbol{k}} \zeta^{\star}_S(\boldsymbol{k}) A^{\operatorname{dep}\boldsymbol{k}} W^{|\boldsymbol{k}|} = \frac{\sin \pi W}{\pi W} \cdot \frac{\pi(1+A)W}{\sin \pi(1+A)W}$$

in $\mathcal{Z}[A][[W]]$.

Proof. Set $x = 1$ and $y = -1$ in Proposition 3.6 and use Lemma 1.19. $\qquad\square$

Corollary 3.10. *If r is a nonnegative integer and w is an integer with $w \geq r$, then*

$$\sum_{\substack{|\boldsymbol{k}|=w \\ \operatorname{dep}\boldsymbol{k}=r}} \zeta_S(\boldsymbol{k}), \sum_{\substack{|\boldsymbol{k}|=w \\ \operatorname{dep}\boldsymbol{k}=r}} \zeta^{\star}_S(\boldsymbol{k}) \begin{cases} = 0 & \text{if } w \text{ is odd;} \\ \in \mathbb{Q}\pi^w & \text{if } w \text{ is even.} \end{cases}$$

Proof. Since

$$\frac{\sin \pi W}{\pi W}, \frac{\pi W}{\sin \pi W} \in \mathbb{Q}[[\pi^2 W^2]],$$

the corollary is immediate from Proposition 3.9. $\qquad\square$

Example 3.11. Setting $x = 1$ and $y = -1$ in Example 3.8 gives

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=0}} \zeta_S(\boldsymbol{k}) = 0,$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=1}} \zeta_S(\boldsymbol{k}) = 2\zeta(4) = \frac{\pi^4}{45},$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=2}} \zeta_S(\boldsymbol{k}) = -3\zeta(4) + 2\zeta(2)^2 = \frac{\pi^4}{45},$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=3}} \zeta_S(\boldsymbol{k}) = 2\zeta(4) - 2\zeta(2)^2 = -\frac{\pi^4}{30},$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=4}} \zeta_S(\boldsymbol{k}) = -\frac{\zeta(4)}{2} + \frac{\zeta(2)^2}{2} = \frac{\pi^4}{120}$$

and

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=0}} \zeta_S^\star(\boldsymbol{k}) = 0,$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=1}} \zeta_S^\star(\boldsymbol{k}) = 2\zeta(4) = \frac{\pi^4}{45},$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=2}} \zeta_S^\star(\boldsymbol{k}) = 3\zeta(4) + 2\zeta(2)^2 = \frac{4\pi^4}{45},$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=3}} \zeta_S^\star(\boldsymbol{k}) = 2\zeta(4) + 2\zeta(2)^2 = \frac{7\pi^4}{90},$$

$$\sum_{\substack{|\boldsymbol{k}|=4 \\ \mathrm{dep}\,\boldsymbol{k}=4}} \zeta_S^\star(\boldsymbol{k}) = \frac{\zeta(4)}{2} + \frac{\zeta(2)^2}{2} = \frac{7\pi^4}{360}.$$

Remark 3.12. Using the explicit expansion

$$\frac{\sin \pi W}{\pi W} = \sum_{m=0}^\infty \frac{(-1)^m}{(2m+1)!} \pi^{2m} W^{2m}, \quad \frac{\pi W}{\sin \pi W} = \sum_{m=0}^\infty \frac{(-1)^m (2 - 2^{2m}) B_{2m}}{(2m)!} \pi^{2m} W^{2m},$$

where B_{2m} are the Bernoulli numbers, we may explicitly write the sums in Corollary 3.10 in terms of the Bernoulli numbers, though such explicit formulas are not as nice as the formal power series formulas given in Proposition 3.9.

4. Schur multiple zeta values of anti-hook type

4.1. **Schur multiple zeta values of anti-hook type.** When investigating the relationship between the sum formulas for multiple zeta(-star) values and symmetric multiple zeta(-star) values, we find it necessary to use the Schur multiple zeta

values (defined by Nakasuji, Phuksuwan, and Yamasaki [9]) of anti-hook type. If $(k_1, \ldots, k_r)$ and $(l_1, \ldots, l_s)$ are indices and a is a positive integer with $a \geq 2$, then the *Schur multiple zeta value of anti-hook type* is defined as

$$\zeta \left(\begin{array}{c} \boxed{\begin{array}{c} k_1 \\ \vdots \\ k_r \end{array}} \\ \boxed{l_1 \ \cdots \ l_s \ a} \end{array} \right) = \sum \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_1^{l_1} \cdots n_s^{l_s} p^a},$$

where the sum runs over all positive integers $m_1, \ldots, m_r, n_1, \ldots, n_s, p$ satisfying $m_1 < \cdots < m_r < p$ and $n_1 \leq \cdots \leq n_s \leq p$. To save space, we write $\begin{pmatrix} k_1, \ldots, k_r \\ l_1, \ldots, l_s \end{pmatrix} ; a \end{pmatrix}$ for what lies between the parentheses in the left-hand side. The Schur multiple zeta values of anti-hook type are a common generalization of the multiple zeta and zeta-star values:

$$\zeta \begin{pmatrix} \boldsymbol{k} \\ \emptyset \end{pmatrix} ; a \end{pmatrix} = \zeta(\boldsymbol{k}, a), \qquad \zeta \begin{pmatrix} \emptyset \\ \boldsymbol{l} \end{pmatrix} ; a \end{pmatrix} = \zeta^\star(\boldsymbol{l}, a)$$

if $\boldsymbol{k}$ and $\boldsymbol{l}$ are indices and $a \geq 2$. An advantage of using the Schur multiple zeta values of anti-hook type is that they allow us to succinctly express the harmonic product of the multiple zeta value and the multiple zeta-star value:

$$\zeta(k_1, \ldots, k_r)\zeta^\star(l_1, \ldots, l_s) = \left(\sum_{1 \leq m_1 < \cdots < m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \right) \left(\sum_{1 \leq n_1 \leq \cdots \leq n_s} \frac{1}{n_1^{l_1} \cdots n_s^{l_s}} \right)$$

$$= \left(\sum_{\substack{1 \leq m_1 < \cdots < m_r \\ 1 \leq n_1 \leq \cdots \leq n_s \\ m_r \geq n_s}} + \sum_{\substack{1 \leq m_1 < \cdots < m_r \\ 1 \leq n_1 \leq \cdots \leq n_s \\ m_r < n_s}} \right) \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_1^{l_1} \cdots n_s^{l_s}}$$

$$= \zeta \begin{pmatrix} k_1, \ldots, k_{r-1} \\ l_1, \ldots, l_s \end{pmatrix} ; k_r \end{pmatrix} + \zeta \begin{pmatrix} k_1, \ldots, k_r \\ l_1, \ldots, l_{s-1} \end{pmatrix} ; l_s \end{pmatrix}$$

if $(k_1, \ldots, k_r)$ and $(l_1, \ldots, l_s)$ are nonempty admissible indices (see [2, Lemma 2.2] for the general formula of the harmonic product of two Schur multiple zeta values). Observe that the Schur multiple zeta value can always be written as a $\mathbb{Q}$-linear combination of multiple zeta values; for example we have

$$\zeta \begin{pmatrix} k \\ l \end{pmatrix} ; a \end{pmatrix} = \zeta(k, l, a) + \zeta(l, k, a) + \zeta(k + l, a) + \zeta(k, l + a).$$

4.2. Elements of $\mathcal{I}$ corresponding to Schur multiple zeta values of anti-hook type.

The observations made in the previous subsection lead us to the following formal definition of $\begin{bmatrix} k_1, \ldots, k_r \\ l_1, \ldots, l_s \end{bmatrix} ; a \end{bmatrix} \in \mathcal{I}$:

Definition 4.1. We define

$$\begin{bmatrix} k_1, \ldots, k_r \\ l_1, \ldots, l_s \end{bmatrix} ; a \end{bmatrix} \in \mathcal{I}$$

for each pair of indices $(k_1, \ldots, k_r)$ and $(l_1, \ldots, l_s)$ and each positive integer a so that the following properties are fulfilled:

(1) $\begin{bmatrix} k_1, \ldots, k_r \\ \emptyset \end{bmatrix} ; a \end{bmatrix} = [k_1, \ldots, k_r, a]$ and $\begin{bmatrix} \emptyset \\ l_1, \ldots, l_s \end{bmatrix} ; a \end{bmatrix} = [l_1, \ldots, l_s, a]^\star$ whenever $(k_1, \ldots, k_r)$ and $(l_1, \ldots, l_s)$ are indices and a is a positive integer;

(2) $\begin{bmatrix} k_1, \ldots, k_{r-1} \\ l_1, \ldots, l_s \end{bmatrix} ; k_r \end{bmatrix} + \begin{bmatrix} k_1, \ldots, k_r \\ l_1, \ldots, l_{s-1} \end{bmatrix} ; l_s \end{bmatrix} = [k_1, \ldots, k_r] * [l_1, \ldots, l_s]^\star$ whenever $(k_1, \ldots, k_r)$ and $(l_1, \ldots, l_s)$ are nonempty indices.

Observe that the definition above does indeed uniquely determine $\begin{bmatrix} k_1, \ldots, k_r \\ l_1, \ldots, l_s \end{bmatrix} ; a \end{bmatrix} \in$ $\mathcal{I}$ as the following example illustrates. The required properties imply that

$$\begin{bmatrix} k_1, k_2, k_3 \\ \emptyset \end{bmatrix} ; k_4 \end{bmatrix} = [k_1, k_2, k_3, k_4],$$

$$\begin{bmatrix} k_1, k_2, k_3 \\ \emptyset \end{bmatrix} ; k_4 \end{bmatrix} + \begin{bmatrix} k_1, k_2 \\ k_4 \end{bmatrix} ; k_3 \end{bmatrix} = [k_1, k_2, k_3] * [k_4]^\star,$$

$$\begin{bmatrix} k_1, k_2 \\ k_4 \end{bmatrix} ; k_3 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_4, k_3 \end{bmatrix} ; k_2 \end{bmatrix} = [k_1, k_2] * [k_4, k_3]^\star,$$

$$\begin{bmatrix} k_1 \\ k_4, k_3 \end{bmatrix} ; k_2 \end{bmatrix} + \begin{bmatrix} \emptyset \\ k_4, k_3, k_2 \end{bmatrix} ; k_1 \end{bmatrix} = [k_1] * [k_4, k_3, k_2]^\star,$$

$$\begin{bmatrix} \emptyset \\ k_4, k_3, k_2 \end{bmatrix} ; k_1 \end{bmatrix} = [k_4, k_3, k_2, k_1]^\star.$$

Although the requirements are superfluous (there are 4 unknowns and 5 equations in the example above), the third remark after Theorem 2.1 shows that they are compatible.

Proposition 4.2. *If $(k_1, \ldots, k_r)$ and $(l_1, \ldots, l_s)$ are indices and a is a positive integer, then we have*

$$\tilde{S}\left(\begin{bmatrix} k_1, \ldots, k_r \\ l_1, \ldots, l_s \end{bmatrix} ; a \end{bmatrix} \right) = (-1)^{r+s+1} \begin{bmatrix} l_1, \ldots, l_s \\ k_1, \ldots, k_r \end{bmatrix} ; a \end{bmatrix}.$$

Proof. Since $\tilde{S}$ is an involution, the proposition is equivalent to showing that

$$\begin{bmatrix} k_1, \ldots, k_r \\ l_1, \ldots, l_s \end{bmatrix} ; a \end{bmatrix} = (-1)^{r+s+1} \tilde{S}\left(\begin{bmatrix} l_1, \ldots, l_s \\ k_1, \ldots, k_r \end{bmatrix} ; a \end{bmatrix} \right).$$

The map $((k_1, \ldots, k_r), (l_1, \ldots, l_s), a) \mapsto (-1)^{r+s+1} \tilde{S}\left(\begin{bmatrix} l_1, \ldots, l_s \\ k_1, \ldots, k_r \end{bmatrix} ; a \end{bmatrix} \right)$ satisfies the properties in Definition 4.1 because

$$(-1)^{r+1} \tilde{S}\left(\begin{bmatrix} \emptyset \\ k_1, \ldots, k_r \end{bmatrix} ; a \end{bmatrix} \right) = (-1)^{r+1} \tilde{S}([k_1, \ldots, k_r, a]^\star) = [k_1, \ldots, k_r, a],$$

$$(-1)^{s+1} \tilde{S}\left(\begin{bmatrix} l_1, \ldots, l_s \\ \emptyset \end{bmatrix} ; a \end{bmatrix} \right) = (-1)^{s+1} \tilde{S}([l_1, \ldots, l_s, a]) = [l_1, \ldots, l_s, a]^\star$$

and

$$(-1)^{r+s}\tilde{S}\left(\begin{bmatrix} l_1,\dots,l_s \\ k_1,\dots,k_{r-1} \end{bmatrix}; k_r\right) + (-1)^{r+s}\tilde{S}\left(\begin{bmatrix} l_1,\dots,l_{s-1} \\ k_1,\dots,k_r \end{bmatrix}; l_s\right)$$

$$= (-1)^{r+s}\tilde{S}\left(\begin{bmatrix} l_1,\dots,l_s \\ k_1,\dots,k_{r-1} \end{bmatrix}; k_r + \begin{bmatrix} l_1,\dots,l_{s-1} \\ k_1,\dots,k_r \end{bmatrix}; l_s\right)$$

$$= (-1)^{r+s}\tilde{S}([l_1,\dots,l_s] * [k_1,\dots,k_r]^\star)$$

$$= (-1)^{r+s}\tilde{S}([l_1,\dots,l_s]) * \tilde{S}([k_1,\dots,k_r]^\star)$$

$$= [l_1,\dots,l_s]^\star * [k_1,\dots,k_r],$$

from which the proposition follows. $\square$

Lemma 4.3. *If k and l are indices and a is a positive integer, then we have*

$$\sum_{j=0}^{s} (-1)^j [k, a, l_j] * [\overleftarrow{l^j}]^\star = \begin{bmatrix} k \\ \overleftarrow{l} \end{bmatrix}; a \end{bmatrix},$$

where $s = \operatorname{dep} l$.

Proof. If we write $l = (l_1,\dots,l_s)$, then Definition 4.1 implies that

$$[k, a, l_j] * [\overleftarrow{l^j}]^\star = \begin{bmatrix} k, a, l_{j-1} \\ \overleftarrow{l^j} \end{bmatrix}; l_j \end{bmatrix} + \begin{bmatrix} k, a, l_j \\ \overleftarrow{l^{j+1}} \end{bmatrix}; l_{j+1} \end{bmatrix}$$

for $j = 0,\dots,s$, where in the right-hand side we understand that the first term is $\begin{bmatrix} k \\ \overleftarrow{l} \end{bmatrix}; a \end{bmatrix}$ if $j = 0$ and that the second term is 0 if $j = s$. This immediately implies the lemma. $\square$

If $k = (k_1,\dots,k_r)$ is an index and i and i' are integers with $0 \le i \le i' \le r$, then we write $k_{i'}^i = (k_{i+1},\dots,k_{i'})$.

Lemma 4.4. *If k and l are indices and a is a positive integer, then we have*

$$\sum_{j=0}^{s} (-1)^j \begin{bmatrix} k \\ \overleftarrow{l_j} \end{bmatrix}; a \end{bmatrix} * [l^j] = [k, a, l],$$

where $s = \operatorname{dep} l$.

Proof. Lemma 4.3 shows that

$$\sum_{j=0}^{s} (-1)^j \begin{bmatrix} k \\ \overleftarrow{l_j} \end{bmatrix}; a \end{bmatrix} * [l^j] = \sum_{j=0}^{s} (-1)^j \left(\sum_{j'=0}^{j} (-1)^{j'} [k, a, l_{j'}] * [\overleftarrow{l_j^{j'}}]^\star \right) * [l^j]$$

$$= \sum_{j'=0}^{s} (-1)^{s-j'} [k, a, l_{j'}] * \left(\sum_{j=j'}^{s} (-1)^{s-j} [\overleftarrow{l_j^{j'}}]^\star * [l^j] \right)$$

$$= [k, a, l],$$

where the last equality follows from the observation that the third remark after Theorem 2.1 implies that the sum in the round brackets vanishes unless $j' = s$, in which case it is equal to $[\emptyset]$. $\square$

The following lemma will be the key to explaining the relationship between the sum formulas for multiple zeta(-star) values and symmetric multiple zeta(-star) values and also to proving our main theorem:

Lemma 4.5. *If $\boldsymbol{k}$ and $\boldsymbol{l}$ are indices and a is a positive integer, then we have*

$$[\boldsymbol{k}, a, \boldsymbol{l}]_{x,y} = \sum_{i=0}^{r}(-1)^{r-i}\begin{bmatrix} \overleftarrow{\boldsymbol{l}} \\ \boldsymbol{k}^i \end{bmatrix};a\end{bmatrix} y^{|\boldsymbol{k}^i|+a+|\boldsymbol{l}|} * [\boldsymbol{k}_i]_{x,y} + \sum_{j=0}^{s}(-1)^{j}\begin{bmatrix} \boldsymbol{k} \\ \overleftarrow{\boldsymbol{l}_j} \end{bmatrix};a\end{bmatrix} x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_j|} * [\boldsymbol{l}^j]_{x,y},$$

$$[\boldsymbol{k}, a, \boldsymbol{l}]_{x,y}^{\star} = \sum_{i=0}^{r}(-1)^{r-i}\begin{bmatrix} \boldsymbol{k}^i \\ \overleftarrow{\boldsymbol{l}} \end{bmatrix};a\end{bmatrix} y^{|\boldsymbol{k}^i|+a+|\boldsymbol{l}|} * [\boldsymbol{k}_i]_{x,y}^{\star} + \sum_{j=0}^{s}(-1)^{j}\begin{bmatrix} \overleftarrow{\boldsymbol{l}_j} \\ \boldsymbol{k} \end{bmatrix};a\end{bmatrix} x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_j|} * [\boldsymbol{l}^j]_{x,y}^{\star},$$

where $r = \operatorname{dep} \boldsymbol{k}$ and $s = \operatorname{dep} \boldsymbol{l}$.

Proof. The first identity implies the second because

$$[\boldsymbol{k}, a, \boldsymbol{l}]_{x,y}^{\star} = (-1)^{r+s+1}\tilde{S}([\boldsymbol{k}, a, \boldsymbol{l}]_{x,y})$$

$$= (-1)^{r+s+1}\sum_{i=0}^{r}(-1)^{r-i}\tilde{S}\left(\begin{bmatrix} \overleftarrow{\boldsymbol{l}} \\ \boldsymbol{k}^i \end{bmatrix};a\end{bmatrix}\right) y^{|\boldsymbol{k}^i|+a+|\boldsymbol{l}|} * \tilde{S}([\boldsymbol{k}_i]_{x,y})$$

$$+ (-1)^{r+s+1}\sum_{j=0}^{s}(-1)^{j}\tilde{S}\left(\begin{bmatrix} \boldsymbol{k} \\ \overleftarrow{\boldsymbol{l}_j} \end{bmatrix};a\end{bmatrix}\right) x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_j|} * \tilde{S}([\boldsymbol{l}^j]_{x,y})$$

$$= (-1)^{r+s+1}\sum_{i=0}^{r}(-1)^{r-i}(-1)^{r-i+s+1}\begin{bmatrix} \boldsymbol{k}^i \\ \overleftarrow{\boldsymbol{l}} \end{bmatrix};a\end{bmatrix} y^{|\boldsymbol{k}^i|+a+|\boldsymbol{l}|} * (-1)^{i}[\boldsymbol{k}_i]_{x,y}^{\star}$$

$$+ (-1)^{r+s+1}\sum_{j=0}^{s}(-1)^{j}(-1)^{r+j+1}\begin{bmatrix} \overleftarrow{\boldsymbol{l}_j} \\ \boldsymbol{k} \end{bmatrix};a\end{bmatrix} x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_j|} * (-1)^{s-j}[\boldsymbol{l}^j]_{x,y}^{\star}$$

$$= \sum_{i=0}^{r}(-1)^{r-i}\begin{bmatrix} \boldsymbol{k}^i \\ \overleftarrow{\boldsymbol{l}} \end{bmatrix};a\end{bmatrix} y^{|\boldsymbol{k}^i|+a+|\boldsymbol{l}|} * [\boldsymbol{k}_i]_{x,y}^{\star} + \sum_{j=0}^{s}(-1)^{j}\begin{bmatrix} \overleftarrow{\boldsymbol{l}_j} \\ \boldsymbol{k} \end{bmatrix};a\end{bmatrix} x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_j|} * [\boldsymbol{l}^j]_{x,y}^{\star}.$$

Lemma 4.4 shows that

$$
\begin{aligned}
[\boldsymbol{k},a,\boldsymbol{l}]_{x,y} &= \sum_{i=0}^{r}[\boldsymbol{k}_i]*[\overleftarrow{\boldsymbol{l}},a,\overleftarrow{\boldsymbol{k}^i}]x^{|\boldsymbol{k}_i|}y^{|\boldsymbol{k}^i|+a+|\boldsymbol{l}|}+\sum_{j=0}^{s}[\boldsymbol{k},a,\boldsymbol{l}_j]*[\overleftarrow{\boldsymbol{l}^j}]x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_j|}y^{|\boldsymbol{l}^j|}\\
&=\sum_{i=0}^{r}[\boldsymbol{k}_i]*\left(\sum_{i'=i}^{r}(-1)^{r-i'}\left[\begin{array}{c}\overleftarrow{\boldsymbol{l}}\\\boldsymbol{k}^{i'}\end{array};a\right]*[\overleftarrow{\boldsymbol{k}^i_{i'}}]\right)x^{|\boldsymbol{k}_i|}y^{|\boldsymbol{k}^i|+a+|\boldsymbol{l}|}\\
&\quad+\sum_{j=0}^{s}\left(\sum_{j'=0}^{j}(-1)^{j'}\left[\begin{array}{c}\boldsymbol{k}\\\overleftarrow{\boldsymbol{l}}_{j'}\end{array};a\right]*[\boldsymbol{l}^{j'}_j]\right)*[\overleftarrow{\boldsymbol{l}^j}]x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_j|}y^{|\boldsymbol{l}^j|}\\
&=\sum_{i'=0}^{r}(-1)^{r-i'}\left[\begin{array}{c}\overleftarrow{\boldsymbol{l}}\\\boldsymbol{k}^{i'}\end{array};a\right]y^{|\boldsymbol{k}^{i'}|+a+|\boldsymbol{l}|}*\left(\sum_{i=0}^{i'}[\boldsymbol{k}_i]*[\overleftarrow{\boldsymbol{k}^i_{i'}}]x^{|\boldsymbol{k}_i|}y^{|\boldsymbol{k}^i_{i'}|}\right)\\
&\quad+\sum_{j'=0}^{s}(-1)^{j'}\left[\begin{array}{c}\boldsymbol{k}\\\overleftarrow{\boldsymbol{l}}_{j'}\end{array};a\right]x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_{j'}|}*\left(\sum_{j=j'}^{s}[\boldsymbol{l}^{j'}_j]*[\overleftarrow{\boldsymbol{l}^j}]x^{|\boldsymbol{l}^{j'}_j|}y^{|\boldsymbol{l}^j|}\right)\\
&=\sum_{i'=0}^{r}(-1)^{r-i'}\left[\begin{array}{c}\overleftarrow{\boldsymbol{l}}\\\boldsymbol{k}^{i'}\end{array};a\right]y^{|\boldsymbol{k}^{i'}|+a+|\boldsymbol{l}|}*[\boldsymbol{k}_{i'}]_{x,y}\\
&\quad+\sum_{j'=0}^{s}(-1)^{j'}\left[\begin{array}{c}\boldsymbol{k}\\\overleftarrow{\boldsymbol{l}}_{j'}\end{array};a\right]x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_{j'}|}*[\boldsymbol{l}^{j'}]_{x,y},
\end{aligned}
$$

which completes the proof. $\qquad\square$

4.3. Sum formula for Schur multiple zeta values of anti-hook type.

Theorem 4.6 (Bachmann, Kadota, Suzuki, Yamamoto, and Yamasaki [1]). *If r and s are nonnegative integers and w is an integer with $w \geq r + s + 2$, then*

$$
\sum_{\substack{k_1+\cdots+k_r+a+l_1+\cdots+l_s=w\\k_1,\ldots,k_r,l_1,\ldots,l_s\geq1\\a\geq2}}\zeta\left(\begin{array}{c}k_1,\ldots,k_r\\l_1,\ldots,l_s\end{array};a\right)=\binom{w-1}{s}\zeta(w).
$$

Remark 4.7. This theorem is a common generalization of the identities in Theorem 1.1.

This theorem can be rephrased as follows:

Proposition 4.8. *We have*

$$
\sum_{\substack{\boldsymbol{k},\boldsymbol{l}\\a\geq2}}\zeta\left(\begin{array}{c}\boldsymbol{k}\\\boldsymbol{l}\end{array};a\right)A^{\operatorname{dep}\boldsymbol{k}}B^{\operatorname{dep}\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}=\frac{W}{1-A}(\psi_1((1+B)W)-\psi_1((A+B)W))
$$

in $\mathcal{Z}[A,B][[W]]$.

Proof. Theorem 4.6 shows that

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l}\\a\geq 2}}\zeta\left(\begin{array}{c}\boldsymbol{k}\\\boldsymbol{l}\end{array};a\right)A^{\operatorname{dep}\boldsymbol{k}}B^{\operatorname{dep}\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}=\sum_{\substack{r,s\geq 0\\w\geq r+s+2}}\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w\\\operatorname{dep}\boldsymbol{k}=r,\operatorname{dep}\boldsymbol{l}=s\\a\geq 2}}\zeta\left(\begin{array}{c}\boldsymbol{k}\\\boldsymbol{l}\end{array};a\right)A^{r}B^{s}W^{w}$$

$$=\sum_{\substack{r,s\geq 0\\w\geq r+s+2}}\binom{w-1}{s}\zeta(w)A^{r}B^{s}W^{w}$$

$$=\sum_{w=2}^{\infty}\sum_{s=0}^{w-2}\sum_{r=0}^{w-s-2}\binom{w-1}{s}A^{r}B^{s}\zeta(w)W^{w}$$

$$=\sum_{w=2}^{\infty}\sum_{s=0}^{w-2}\binom{w-1}{s}\frac{1-A^{w-s-1}}{1-A}B^{s}\zeta(w)W^{w}$$

$$=\sum_{w=2}^{\infty}\frac{(1+B)^{w-1}-(A+B)^{w-1}}{1-A}\zeta(w)W^{w}$$

$$=\frac{W}{1-A}(\psi_{1}((1+B)W)-\psi_{1}((A+B)W)),$$

as required. □

Remark 4.9. If we define

$$\psi_{1,\mathcal{I}}(W)=\sum_{k=2}^{\infty}[k]W^{k-1}\in\mathcal{I}[[W]],$$

then

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l}\\a\geq 2}}\left[\begin{array}{c}\boldsymbol{k}\\\boldsymbol{l}\end{array};a\right]A^{\operatorname{dep}\boldsymbol{k}}B^{\operatorname{dep}\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}\neq\frac{W}{1-A}(\psi_{1,\mathcal{I}}((1+B)W)-\psi_{1,\mathcal{I}}((A+B)W))$$

in $\mathcal{I}[A,B][[W]]$ because, for example, the coefficient of W^{3} in the left-hand side is

$$\left[\begin{array}{c}\emptyset\\\emptyset\end{array};3\right]+\left[\begin{array}{c}1\\\emptyset\end{array};2\right]A+\left[\begin{array}{c}\emptyset\\1\end{array};2\right]B=[3]+[1,2]A+[1,2]^{\star}B,$$

whereas that in the right-hand side is

$$\frac{[3]}{1-A}((1+B)^{2}-(A+B)^{2})=[3]+[3]A+2[3]B.$$

4.4. Relationship between the sum formulas for multiple zeta values and symmetric multiple zeta values.

The following proposition somehow explains the similarity between the sum formulas for multiple zeta(-star) values and symmetric multiple zeta(-star) values:

Proposition 4.10. *If $\boldsymbol{k}$ and $\boldsymbol{l}$ are indices and a is an integer with $a\geq 2$, then we have*

$$\zeta_{S}(\boldsymbol{k},a,\boldsymbol{l})=\sum_{i=0}^{r}(-1)^{r-i}\zeta\left(\begin{array}{c}\overleftarrow{\boldsymbol{l}}\\\boldsymbol{k}^{i}\end{array};a\right)(-1)^{|\boldsymbol{k}^{i}|+a+|\boldsymbol{l}|}\zeta_{S}(\boldsymbol{k}_{i})+\sum_{j=0}^{s}(-1)^{j}\zeta\left(\begin{array}{c}\boldsymbol{k}\\\overleftarrow{\boldsymbol{l}_{j}}\end{array};a\right)\zeta_{S}(\boldsymbol{l}^{j}),$$

$$\zeta_{S}^{\star}(\boldsymbol{k},a,\boldsymbol{l})=\sum_{i=0}^{r}(-1)^{r-i}\zeta\left(\begin{array}{c}\boldsymbol{k}^{i}\\\overleftarrow{\boldsymbol{l}}\end{array};a\right)(-1)^{|\boldsymbol{k}^{i}|+a+|\boldsymbol{l}|}\zeta_{S}^{\star}(\boldsymbol{k}_{i})+\sum_{j=0}^{s}(-1)^{j}\zeta\left(\begin{array}{c}\overleftarrow{\boldsymbol{l}_{j}}\\\boldsymbol{k}\end{array};a\right)\zeta_{S}^{\star}(\boldsymbol{l}^{j}),$$

where $r = \operatorname{dep} \boldsymbol{k}$ and $s = \operatorname{dep} \boldsymbol{l}$.

Proof. Apply Z to the identities in Lemma 4.5, and set $x = 1$ and $y = -1$. $\square$

We now deduce Theorem 1.3 from Theorem 4.6, which is a generalization of Theorem 1.1, with the aid of Proposition 4.10. Let r and s be nonnegative integers and let w be an integer with $w \geq r + s + 2$. Then by summing the identities in Proposition 4.10 over all indices $\boldsymbol{k}$ and $\boldsymbol{l}$ and all integers $a \geq 2$ satisfying $\operatorname{dep} \boldsymbol{k} = r$, $\operatorname{dep} \boldsymbol{l} = s$, and $|\boldsymbol{k}| + a + |\boldsymbol{l}| = w$ and by using Theorem 4.6 and the fact that any symmetric sum of ζ_S (or $\zeta_S^\star$) of depth greater than 0 is 0 modulo $\zeta(2)\mathcal{Z}$ (see [8, Theorem 1.1] for ζ_S; for $\zeta_S^\star$, observe that when each $\zeta_S^\star$ is expressed as a sum of ζ_S, a symmetric sum of $\zeta_S^\star$ becomes a symmetric sum of ζ_S), we obtain

$$
\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \operatorname{dep}\boldsymbol{k}=r, \operatorname{dep}\boldsymbol{l}=s \\ a\geq 2}} \zeta_S(\boldsymbol{k}, a, \boldsymbol{l})
$$

$$
= \sum_{i=0}^{r} (-1)^{r-i} \sum_{\substack{w_1+w_2=w \\ w_1\geq 2, w_2\geq 0}} \left(\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w_1 \\ \operatorname{dep}\boldsymbol{k}=r-i, \operatorname{dep}\boldsymbol{l}=s \\ a\geq 2}} \zeta\left(\begin{array}{c} \boldsymbol{l} \\ \boldsymbol{k} \end{array}; a \right) \right) (-1)^{w_1} \left(\sum_{\substack{|\boldsymbol{k}|=w_2 \\ \operatorname{dep}\boldsymbol{k}=i}} \zeta_S(\boldsymbol{k}) \right)
$$

$$
+ \sum_{j=0}^{s} (-1)^{j} \sum_{\substack{w_1+w_2=w \\ w_1\geq 2, w_2\geq 0}} \left(\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w_1 \\ \operatorname{dep}\boldsymbol{k}=r, \operatorname{dep}\boldsymbol{l}=j \\ a\geq 2}} \zeta\left(\begin{array}{c} \boldsymbol{k} \\ \boldsymbol{l} \end{array}; a \right) \right) \left(\sum_{\substack{|\boldsymbol{l}|=w_2 \\ \operatorname{dep}\boldsymbol{l}=s-j}} \zeta_S(\boldsymbol{l}) \right)
$$

$$
\equiv (-1)^{r} \left(\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \operatorname{dep}\boldsymbol{k}=r, \operatorname{dep}\boldsymbol{l}=s \\ a\geq 2}} \zeta\left(\begin{array}{c} \boldsymbol{l} \\ \boldsymbol{k} \end{array}; a \right) \right) (-1)^{w} + (-1)^{s} \left(\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \operatorname{dep}\boldsymbol{k}=r, \operatorname{dep}\boldsymbol{l}=s \\ a\geq 2}} \zeta\left(\begin{array}{c} \boldsymbol{k} \\ \boldsymbol{l} \end{array}; a \right) \right)
$$

$$
= (-1)^{r} \binom{w-1}{r} \zeta(w)(-1)^{w} + (-1)^{s} \binom{w-1}{s} \zeta(w)
$$

$$
\equiv \left(-(-1)^{r} \binom{w-1}{r} + (-1)^{s} \binom{w-1}{s} \right) \zeta(w)
$$

and

$$\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \mathrm{dep}\,\boldsymbol{k}=r,\mathrm{dep}\,\boldsymbol{l}=s \\ a\geq 2}} \zeta_S^\star(\boldsymbol{k},a,\boldsymbol{l})$$

$$= \sum_{i=0}^{r}(-1)^{r-i}\sum_{\substack{w_1+w_2=w \\ w_1\geq 2,w_2\geq 0}}\left(\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w_1 \\ \mathrm{dep}\,\boldsymbol{k}=r-i,\mathrm{dep}\,\boldsymbol{l}=s \\ a\geq 2}}\zeta\left(\begin{array}{c}\boldsymbol{k}\\\boldsymbol{l}\end{array};a\right)\right)(-1)^{w_1}\left(\sum_{\substack{|\boldsymbol{k}|=w_2 \\ \mathrm{dep}\,\boldsymbol{k}=i}}\zeta_S^\star(\boldsymbol{k})\right)$$

$$+\sum_{j=0}^{s}(-1)^{j}\sum_{\substack{w_1+w_2=w \\ w_1\geq 2,w_2\geq 0}}\left(\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w_1 \\ \mathrm{dep}\,\boldsymbol{k}=r,\mathrm{dep}\,\boldsymbol{l}=j \\ a\geq 2}}\zeta\left(\begin{array}{c}\boldsymbol{l}\\\boldsymbol{k}\end{array};a\right)\right)\left(\sum_{\substack{|\boldsymbol{l}|=w_2 \\ \mathrm{dep}\,\boldsymbol{l}=s-j}}\zeta_S^\star(\boldsymbol{l})\right)$$

$$\equiv (-1)^{r}\left(\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \mathrm{dep}\,\boldsymbol{k}=r,\mathrm{dep}\,\boldsymbol{l}=s \\ a\geq 2}}\zeta\left(\begin{array}{c}\boldsymbol{k}\\\boldsymbol{l}\end{array};a\right)\right)(-1)^{w}+(-1)^{s}\left(\sum_{\substack{|\boldsymbol{k}|+a+|\boldsymbol{l}|=w \\ \mathrm{dep}\,\boldsymbol{k}=r,\mathrm{dep}\,\boldsymbol{l}=s \\ a\geq 2}}\zeta\left(\begin{array}{c}\boldsymbol{l}\\\boldsymbol{k}\end{array};a\right)\right)$$

$$= (-1)^{r}\binom{w-1}{s}\zeta(w)(-1)^{w}+(-1)^{s}\binom{w-1}{r}\zeta(w)$$

$$\equiv \left((-1)^{s}\binom{w-1}{r}-(-1)^{r}\binom{w-1}{s}\right)\zeta(w)$$

modulo $\zeta(2)\mathcal{Z}$.

4.5. **Proof of our main theorem.** We begin with computing the generating functions

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}}[\boldsymbol{k},a,\boldsymbol{l}]_{x,y}A^{\mathrm{dep}\,\boldsymbol{k}}B^{\mathrm{dep}\,\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|},\ \sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}}[\boldsymbol{k},a,\boldsymbol{l}]_{x,y}^\star A^{\mathrm{dep}\,\boldsymbol{k}}B^{\mathrm{dep}\,\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}\in\mathcal{I}[x,y][A,B][[W]].$$

Since it is unlikely that the generating functions can be written in terms of $\Gamma_{1,\mathcal{I}}(W)$ only, we shall use the generating function

$$F_{\mathcal{I}}(A,B,W)=\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}}\left[\begin{array}{c}\boldsymbol{k}\\\boldsymbol{l}\end{array};a\right]A^{\mathrm{dep}\,\boldsymbol{k}}B^{\mathrm{dep}\,\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}\in\mathcal{I}[A,B][[W]],$$

which appeared in Remark 4.9. Note that

$$\tilde{S}(F_{\mathcal{I}}(A,B,W))=\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}}\tilde{S}\left(\left[\begin{array}{c}\boldsymbol{k}\\\boldsymbol{l}\end{array};a\right]\right)A^{\mathrm{dep}\,\boldsymbol{k}}B^{\mathrm{dep}\,\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$=\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}}(-1)^{\mathrm{dep}\,\boldsymbol{k}+\mathrm{dep}\,\boldsymbol{l}+1}\left[\begin{array}{c}\boldsymbol{l}\\\boldsymbol{k}\end{array};a\right]A^{\mathrm{dep}\,\boldsymbol{k}}B^{\mathrm{dep}\,\boldsymbol{l}}W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$=-F_{\mathcal{I}}(-B,-A,W).$$

Proposition 4.11. *We have*

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} [\boldsymbol{k}, a, \boldsymbol{l}]_{x,y} A^{\operatorname{dep}\boldsymbol{k}} B^{\operatorname{dep}\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} = F_{\mathcal{I}}(B, -A, yW) \frac{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}{\Gamma_{1,\mathcal{I}}(x(1-A)W)\Gamma_{1,\mathcal{I}}(y(1-A)W)}$$

$$+ F_{\mathcal{I}}(A, -B, xW) \frac{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}{\Gamma_{1,\mathcal{I}}(x(1-B)W)\Gamma_{1,\mathcal{I}}(y(1-B)W)},$$

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} [\boldsymbol{k}, a, \boldsymbol{l}]_{x,y}^{\star} A^{\operatorname{dep}\boldsymbol{k}} B^{\operatorname{dep}\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} = F_{\mathcal{I}}(-A, B, yW) \frac{\Gamma_{1,\mathcal{I}}(x(1+A)W)\Gamma_{1,\mathcal{I}}(y(1+A)W)}{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}$$

$$+ F_{\mathcal{I}}(-B, A, xW) \frac{\Gamma_{1,\mathcal{I}}(x(1+B)W)\Gamma_{1,\mathcal{I}}(y(1+B)W)}{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}$$

in $\mathcal{I}[x,y][A,B][[W]]$.

Proof. The first identity implies the second because

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} [\boldsymbol{k}, a, \boldsymbol{l}]_{x,y}^{\star} A^{\operatorname{dep}\boldsymbol{k}} B^{\operatorname{dep}\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= -\tilde{S} \left(\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a \geq 2}} [\boldsymbol{k}, a, \boldsymbol{l}]_{x,y} (-A)^{\operatorname{dep}\boldsymbol{k}} (-B)^{\operatorname{dep}\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} \right)$$

$$= -\tilde{S} \left(F_{\mathcal{I}}(-B, A, yW) \frac{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}{\Gamma_{1,\mathcal{I}}(x(1+A)W)\Gamma_{1,\mathcal{I}}(y(1+A)W)} \right.$$

$$\left. + F_{\mathcal{I}}(-A, B, xW) \frac{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}{\Gamma_{1,\mathcal{I}}(x(1+B)W)\Gamma_{1,\mathcal{I}}(y(1+B)W)} \right)$$

$$= F_{\mathcal{I}}(-A, B, yW) \frac{\Gamma_{1,\mathcal{I}}(x(1+A)W)\Gamma_{1,\mathcal{I}}(y(1+A)W)}{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)} + F_{\mathcal{I}}(-B, A, xW) \frac{\Gamma_{1,\mathcal{I}}(x(1+B)W)\Gamma_{1,\mathcal{I}}(y(1+B)W)}{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}.$$

Lemma 4.5 shows that

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} [\boldsymbol{k},a,\boldsymbol{l}]_{x,y} A^{\operatorname{dep}\boldsymbol{k}} B^{\operatorname{dep}\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= \sum_{r,s=0}^{\infty} \sum_{\substack{\operatorname{dep}\boldsymbol{k}=r,\operatorname{dep}\boldsymbol{l}=s \\ a\geq 2}} [\boldsymbol{k},a,\boldsymbol{l}]_{x,y} A^r B^s W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= \sum_{r,s=0}^{\infty} \sum_{\substack{\operatorname{dep}\boldsymbol{k}=r,\operatorname{dep}\boldsymbol{l}=s \\ a\geq 2}} \sum_{i=0}^{r} (-1)^{r-i} \begin{bmatrix} \overleftarrow{\boldsymbol{l}} \\ \boldsymbol{k}^i \end{bmatrix}\!;a\end{bmatrix} y^{|\boldsymbol{k}^i|+a+|\boldsymbol{l}|} * [\boldsymbol{k}_i]_{x,y} A^r B^s W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$+ \sum_{r,s=0}^{\infty} \sum_{\substack{\operatorname{dep}\boldsymbol{k}=r,\operatorname{dep}\boldsymbol{l}=s \\ a\geq 2}} \sum_{j=0}^{s} (-1)^{j} \begin{bmatrix} \boldsymbol{k} \\ \overleftarrow{\boldsymbol{l}_j} \end{bmatrix}\!;a\end{bmatrix} x^{|\boldsymbol{k}|+a+|\boldsymbol{l}_j|} * [\boldsymbol{l}^j]_{x,y} A^r B^s W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= \left(\sum_{i,s=0}^{\infty} \sum_{\substack{\operatorname{dep}\boldsymbol{k}=i,\operatorname{dep}\boldsymbol{l}=s \\ a\geq 2}} \begin{bmatrix} \boldsymbol{l} \\ \boldsymbol{k} \end{bmatrix}\!;a\end{bmatrix} (-A)^i B^s (yW)^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} \right) \left(\sum_{i=0}^{\infty} \sum_{\operatorname{dep}\boldsymbol{k}=i} [\boldsymbol{k}]_{x,y} A^i W^{|\boldsymbol{k}|} \right)$$

$$+ \left(\sum_{r,j=0}^{\infty} \sum_{\substack{\operatorname{dep}\boldsymbol{k}=r,\operatorname{dep}\boldsymbol{l}=j \\ a\geq 2}} \begin{bmatrix} \boldsymbol{k} \\ \boldsymbol{l} \end{bmatrix}\!;a\end{bmatrix} A^r (-B)^j (xW)^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} \right) \left(\sum_{j=0}^{\infty} \sum_{\operatorname{dep}\boldsymbol{l}=j} [\boldsymbol{l}]_{x,y} B^j W^{|\boldsymbol{l}|} \right)$$

$$= \left(\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} \begin{bmatrix} \boldsymbol{l} \\ \boldsymbol{k} \end{bmatrix}\!;a\end{bmatrix} (-A)^{\operatorname{dep}\boldsymbol{k}} B^{\operatorname{dep}\boldsymbol{l}} (yW)^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} \right) \left(\sum_{\boldsymbol{k}} [\boldsymbol{k}]_{x,y} A^{\operatorname{dep}\boldsymbol{k}} W^{|\boldsymbol{k}|} \right)$$

$$+ \left(\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} \begin{bmatrix} \boldsymbol{k} \\ \boldsymbol{l} \end{bmatrix}\!;a\end{bmatrix} A^{\operatorname{dep}\boldsymbol{k}} (-B)^{\operatorname{dep}\boldsymbol{l}} (xW)^{|\boldsymbol{k}|+a+|\boldsymbol{l}|} \right) \left(\sum_{\boldsymbol{l}} [\boldsymbol{l}]_{x,y} B^{\operatorname{dep}\boldsymbol{l}} W^{|\boldsymbol{l}|} \right)$$

$$= F_{\mathcal{I}}(B,-A,yW)\frac{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}{\Gamma_{1,\mathcal{I}}(x(1-A)W)\Gamma_{1,\mathcal{I}}(y(1-A)W)} + F_{\mathcal{I}}(A,-B,xW)\frac{\Gamma_{1,\mathcal{I}}(xW)\Gamma_{1,\mathcal{I}}(yW)}{\Gamma_{1,\mathcal{I}}(x(1-B)W)\Gamma_{1,\mathcal{I}}(y(1-B)W)}$$

by Proposition 3.4. This completes the proof. $\square$

Theorem 4.12 (Main theorem)**.** *We have*

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} \zeta_{x,y}(\boldsymbol{k},a,\boldsymbol{l}) A^{\mathrm{dep}\,\boldsymbol{k}} B^{\mathrm{dep}\,\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= \frac{yW}{1-B}(\psi_1(y(1-A)W) - \psi_1(y(B-A)W))\frac{\Gamma_1(xW)\Gamma_1(yW)}{\Gamma_1(x(1-A)W)\Gamma_1(y(1-A)W)}$$

$$+ \frac{xW}{1-A}(\psi_1(x(1-B)W) - \psi_1(x(A-B)W))\frac{\Gamma_1(xW)\Gamma_1(yW)}{\Gamma_1(x(1-B)W)\Gamma_1(y(1-B)W)},$$

$$\sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} \zeta_{x,y}^{\star}(\boldsymbol{k},a,\boldsymbol{l}) A^{\mathrm{dep}\,\boldsymbol{k}} B^{\mathrm{dep}\,\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

$$= \frac{yW}{1+A}(\psi_1(y(1+B)W) - \psi_1(y(B-A)W))\frac{\Gamma_1(x(1+A)W)\Gamma_1(y(1+A)W)}{\Gamma_1(xW)\Gamma_1(yW)}$$

$$+ \frac{xW}{1+B}(\psi_1(x(1+A)W) - \psi_1(x(A-B)W))\frac{\Gamma_1(x(1+B)W)\Gamma_1(y(1+B)W)}{\Gamma_1(xW)\Gamma_1(yW)}$$

in $\mathcal{Z}[T][x,y][A,B][[W]]$.

Proof. Apply Z to the identities in Proposition 4.11, and having

$$Z(F_{\mathcal{I}}(A,B,W)) = \sum_{\substack{\boldsymbol{k},\boldsymbol{l} \\ a\geq 2}} \zeta\begin{pmatrix} \boldsymbol{k} \\ \boldsymbol{l} \end{pmatrix}; a\end{pmatrix} A^{\mathrm{dep}\,\boldsymbol{k}} B^{\mathrm{dep}\,\boldsymbol{l}} W^{|\boldsymbol{k}|+a+|\boldsymbol{l}|}$$

in mind, use Proposition 4.8. □

Acknowledgements

References

[1] H. Bachmann, S. Kadota, Y. Suzuki, S. Yamamoto, and Y. Yamasaki, *Sum formulas for zeta values of multiposets*, in preparation.

[2] H. Bachmann and Y. Yamasaki, *Checkerboard style Schur multiple zeta values and odd single zeta values*, Math. Z. **290** (2018), no. 3-4, 1173–1197.

[3] A. Granville, *A decomposition of Riemann's zeta-function*, Analytic number theory (Kyoto, 1996), London Math. Soc. Lecture Note Ser. **247**, Cambridge Univ. Press, Cambridge, 1997, 95–101.

[4] M. Hirose, H. Murahara, and S. Saito, *Polynomial generalization of the regularization theorem for multiple zeta values*, Publ. Res. Inst. Math. Sci. **56** (2020), 207–215.

[5] M. E. Hoffman, *Quasi-shuffle products*, J. Algebraic Combin. **11** (2000), no. 1, 49–68.

[6] K. Ihara, M. Kaneko, and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compositio Math. **142** (2006), 307–338.

[7] M. Kaneko and D. Zagier, *Finite multiple zeta values*, in preparation.

[8] H. Murahara, *A note on finite real multiple zeta values*, Kyushu J. Math. **70** (2016), no. 1, 197–204.

[9] M. Nakasuji, O. Phuksuwan, and Y. Yamasaki, *On Schur multiple zeta functions: A combinatoric generalization of multiple zeta functions*, Adv. Math. **333** (2018), 570–619.

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