Knot Points of Typical Continuous Functions

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(Joint work with David Preiss)

Outline

Part I Background

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Part I
Background

Typical continuous functions

Work in $I := [0,1]$.

$C(I) := \{f : I \longrightarrow \mathbb{R} \mid f\text{: continuous}\}$
equipped with the supremum norm.

Definition

A typical $f \in C(I)$ satisfies a property P
$\overset{\text{def.}}{\Longleftrightarrow}$ $\{f \in C(I) \mid P \text{ holds}\}$ is
residual (comeagre) in $C(I)$.

Example

A typical $f \in C(I)$ is nowhere differentiable.

What can we say about Dini derivatives of a typical $f \in C(I)$?

Dini derivatives

Definition (Dini derivatives)

For $f \in C(I)$ and $x \in I$,

$$D^{+}f(x) := \limsup_{y \searrow x} \frac{f(y) - f(x)}{y - x},$$

$$D_{+}f(x) := \liminf_{y \searrow x} \frac{f(y) - f(x)}{y - x},$$

$$D^{-}f(x) := \limsup_{y \nearrow x} \frac{f(y) - f(x)}{y - x},$$

$$D_{-}f(x) := \liminf_{y \nearrow x} \frac{f(y) - f(x)}{y - x}.$$

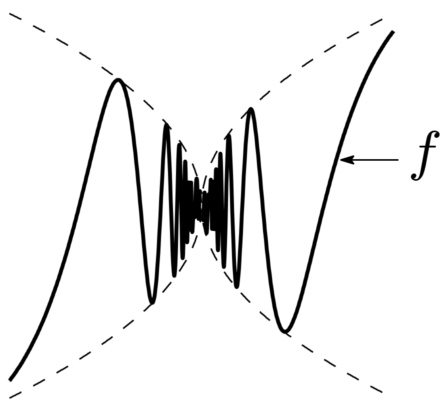
Dini derivatives of a typical $f \in C(I)$

Theorem (Jarník, 1933)

A typical $f \in C(I)$ has the property that

$$D^+ f(x) = D^- f(x) = \infty \text{ and}$$
$$D_+ f(x) = D_- f(x) = -\infty$$

for a.e. $x \in I$.



Such a point x is called a knot point of f.

Knot points of a typical $f \in C(I)$

For $f \in C(I)$,

$$N(f) := \{x \in I \mid x \text{ is NOT a knot point of } f\}.$$

Jarník's theorem asserts that

$$N(f) \text{ is null for a typical } f \in C(I).$$

In what sense of smallness is it true that

$$N(f) \text{ is small for a typical } f \in C(I)?$$

Theorem of Preiss and Zajíček

Theorem (Preiss and Zajíček, unpublished)

For a σ-ideal $\mathcal{I}$ on I, T.F.A.E.:

(1) $N(f) \in \mathcal{I}$ for a typical $f \in C(I)$;

(2) $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$.

Here

$$\mathcal{K} := \{K \subset I \mid K \text{ is closed}\}$$

equipped with the Vietoris topology.

(Hausdorff metric)

Problem

Problem

Characterise those families $\mathcal{A}$ of subsets of I for which

$$N(f) \in \mathcal{A} \text{ for a typical } f \in C(I).$$

Part II
Statement of Main Theorem

An observation

Problem

Characterise $\mathcal{A} \subset \mathcal{P}(I)$ for which

$$N(f) \in \mathcal{A} \text{ for a typical } f \in C(I).$$

It is easy to see that

$$N(f) \text{ is } F_\sigma \text{ for all } f \in C(I).$$

Problem

Characterise $\mathcal{F} \subset \mathcal{F}_\sigma$ for which

$$N(f) \in \mathcal{F} \text{ for a typical } f \in C(I).$$

Problem

Characterise $\mathcal{F} \subset \mathcal{F}_\sigma$ for which

$$N(f) \in \mathcal{F} \text{ for a typical } f \in C(I).$$

It is easy to see that

$$\left\{\mathcal{F} \subset \mathcal{F}_\sigma \;\middle|\; N(f) \in \mathcal{F} \text{ for a typical } f \in C(I)\right\}$$

is a σ-filter of F_σ sets.

Residuality of families of $\mathcal{F}_\sigma$ sets

Proposition (S)

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

(1) $\{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ is residual in $\mathcal{K}^{\mathbb{N}}$.

(2) $\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

Here

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} := \{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid K_1 \subset K_2 \subset \cdots\}.$$

We say that $\mathcal{F}$ is residual in $\mathcal{F}_\sigma$ if the above conditions hold.

Main theorem

Main Theorem

If $\mathcal{F}$ is residual in $\mathcal{F}_\sigma$,

then $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$.

Conjecture

The converse is also true:

if $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$,

then $\mathcal{F}$ is residual in $\mathcal{F}_\sigma$.

Part III
Outline of Proof

Residuality and Banach-Mazur game

X: a topological space, $S \subset X$.

Players I and II alternately choose a nonempty open set. They must choose a subset of the set chosen in the previous turn.

$$\begin{array}{llllll} \text{I:} & U_1 & & U_2 & & \\ & \supset & \supset & \supset & \supset & \cdots \\ \text{II:} & & V_1 & & V_1 & \end{array}$$

Player II wins iff $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} V_n \subset S$.

Fact

Player II has a winning strategy

$\iff$ S is residual.

Outline of the proof

Let $\mathcal{A} := \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$.

We know that $\mathcal{A}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

We want to show that

$$\left\{ f \in C(I) \;\middle|\; N(f) \in \mathcal{F} \right\}$$

is residual in $C(I)$.

It suffices to show that

$$\mathcal{X} = \left\{ f \in C(I) \;\middle|\; \exists (K_n) \in \mathcal{A} \quad N(f) = \bigcup_{n=1}^{\infty} K_n \right\}$$

is residual in $C(I)$.

Outline of the proof

$\mathcal{A}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$

$\Downarrow$

Player II has a winning strategy
in the Banach-Mazur game for $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow}$

$\Downarrow \leftarrow$ similar to Proposition

Player II has a winning strategy
in another game for $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow}$

$\Downarrow \leftarrow$ difficult

Player II has a winning strategy
in the Banach-Mazur game for $\mathcal{X} \subset C(I)$

$\Downarrow$

$\mathcal{X}$ is residual in $C(I)$