Residuality of Families of $\mathcal{F}_\sigma$ Sets

Shingo SAITO

University College London

http://www.ucl.ac.uk/~ucahssa/

It is known that

a *typical* closed subset of $[0,1]$ is null $\cdots (*)$.

What does *typical* mean?

$\mathcal{K} := \{ K \subset [0,1] \mid K\text{: closed} \}$

equipped with the Hausdorff metric d.

$(d(K,\emptyset) := 1$ if $K \in \mathcal{K} \setminus \{\emptyset\})$

($*$) means

$$\{K \in \mathcal{K} \mid K\text{: null}\} \text{ is } \underline{\text{residual}} \text{ in } \mathcal{K}.$$

$\uparrow$

its complement is of first category

Question:

Is a typical $\mathcal{F}_\sigma$ set null?

$\longrightarrow$ What does *typical* mean in this context?

$\mathcal{F}_\sigma := \{F \subset [0,1] \mid F \text{ is } \mathcal{F}_\sigma\}$

We want to define when $\mathcal{F} \subset \mathcal{F}_\sigma$ is residual.

Simplest idea: to topologise $\mathcal{F}_\sigma$

$\longrightarrow$ does not work

We shall define residuality on $\mathcal{F}_\sigma$ without topologising it.

Look at the surjection

$$\begin{array}{ccc} \mathcal{K}^{\mathbb{N}} & \longrightarrow & \mathcal{F}_\sigma \\ \Cup & & \Cup \\ (K_n) & \longmapsto & \bigcup_{n=1}^{\infty} K_n \end{array}$$

<u>Def</u>

$\mathcal{F} \subset \mathcal{F}_\sigma$ is $\mathcal{K}^{\mathbb{N}}$-*residual* if

$$\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \,\middle|\, \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

is residual in $\mathcal{K}^{\mathbb{N}}$.

Another natural surjection:

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} := \{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid K_1 \subset K_2 \subset \cdots\}$$

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} \longrightarrow \mathcal{F}_\sigma;\ (K_n) \longmapsto \bigcup_{n=1}^{\infty} K_n$$

<u>Def</u>

$\mathcal{F} \subset \mathcal{F}_\sigma$ is $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-*residual* if

$$\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$$

is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

Thm 1

$\mathcal{F} \subset \mathcal{F}_\sigma$ is $\mathcal{K}^{\mathbb{N}}$-residual iff it is $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-residual.

Thm 2

For a σ-ideal $\mathcal{I}$ on $[0,1]$, T.F.A.E.:

(1) $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$
(i.e. a typical closed set belongs to $\mathcal{I}$);

(2) $\mathcal{I} \cap \mathcal{F}_\sigma$ is residual in $\mathcal{F}_\sigma$
(i.e. a typical $\mathcal{F}_\sigma$ set belongs to $\mathcal{I}$).

Proof

Thm 1 $\implies$ Thm 2: easy

We use the *Banach-Mazur game* to prove Thm 1.

Banach-Mazur game

X: metric space, $S \subset X$.

Players ① and ② alternately choose a ball.

They must choose a subset of the ball chosen in the previous turn.

$$\begin{array}{ccccccccc} B_1 & \supset & B_2 & \supset & B_3 & \supset & B_4 & \supset & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ ① & & ② & & ① & & ② & & \end{array}$$

Player ② wins iff $\bigcap_{n=1}^{\infty} B_n \subset S$.

Fact

Player ② has a winning strategy

$\iff S$ is residual in X.

Plan for the proof of Thm 1

$(\mathcal{K}^{\mathbb{N}}_{\nearrow}\text{-residual} \implies \mathcal{K}^{\mathbb{N}}\text{-residual})$

$\mathcal{F} \subset \mathcal{F}_\sigma$: $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-residual

$\mathcal{K}^{\mathbb{N}}_{\nearrow}$-game: $X = \mathcal{K}^{\mathbb{N}}_{\nearrow}$

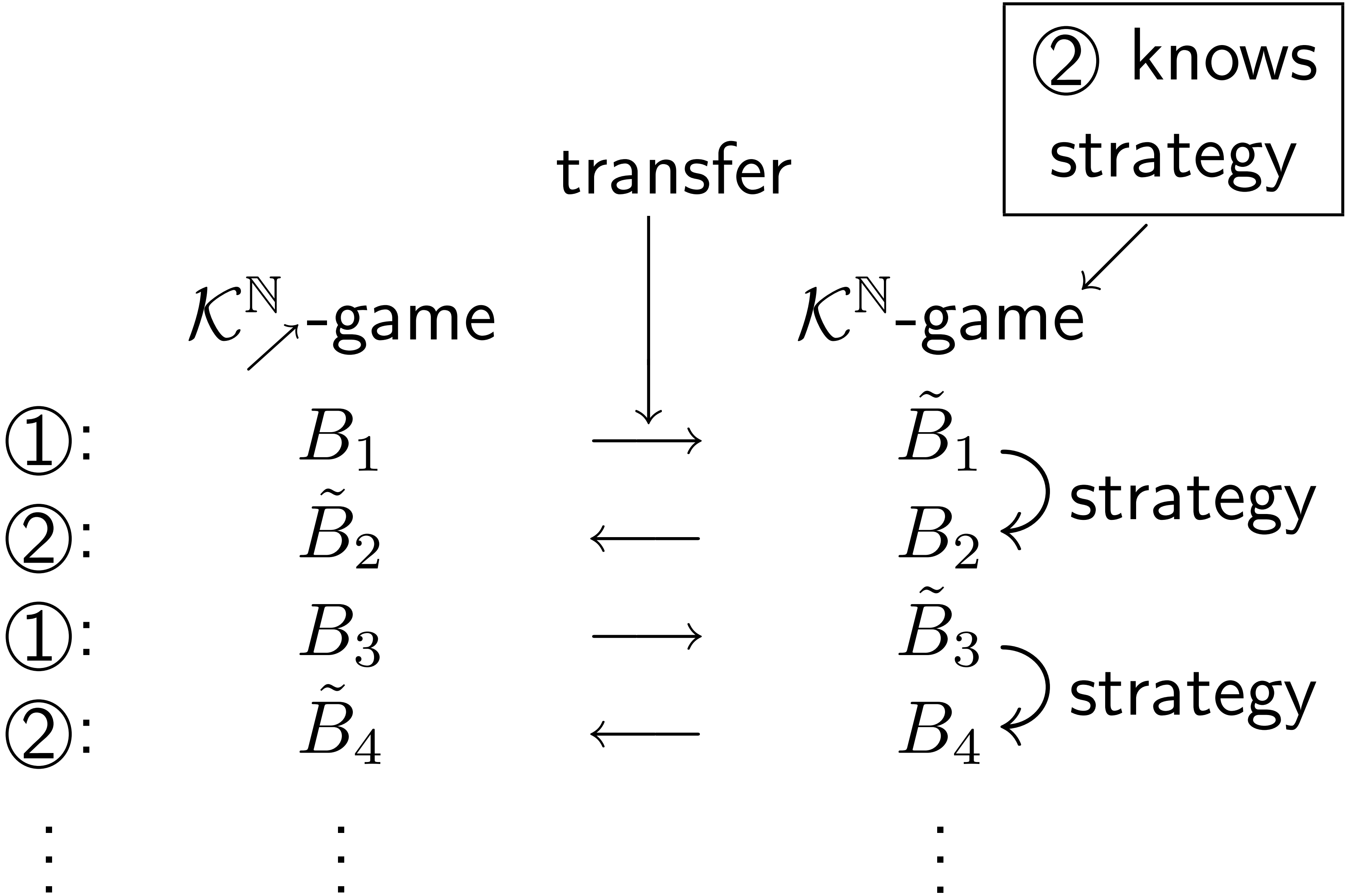
$$S = \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

② knows a winning strategy.

$\mathcal{K}^{\mathbb{N}}$-game: $X = \mathcal{K}^{\mathbb{N}}$

$$S = \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

② looks for a winning strategy.


② knows strategy
transfer
$\mathcal{K}^{\mathbb{N}}_{\nearrow}$-game
$\mathcal{K}^{\mathbb{N}}$-game
①: B_1 $\longrightarrow$ $\tilde{B}_1$
②: $\tilde{B}_2$ $\longleftarrow$ B_2
①: B_3 $\longrightarrow$ $\tilde{B}_3$
②: $\tilde{B}_4$ $\longleftarrow$ B_4
strategy
strategy

How do the transfers go?

The centres matter; the radii do not.

We need an idea.

The simple map

$$\begin{array}{ccc} \mathcal{K}^{\mathbb{N}} & \longrightarrow & \mathcal{K}^{\mathbb{N}}_{\nearrow} \\ \cup & & \cup \\ (K_n) & \longmapsto & (K_1, K_1 \cup K_2, \ldots) \end{array}$$

does not work at all.