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Residuality of Families of $\mathcal{F}_\sigma$ Sets

Abstract

We give two natural definitions of residuality of families of $\mathcal{F}_\sigma$ sets, which turn out to be equivalent. We also introduce the Banach-Mazur game as a strong tool for proving theorems concerning residuality.

1 Introduction

This report is based upon the author's lecture in the Symposium on his preprint [2]. The reader is reminded that the file of the transparencies used in the talk can be found at

`http://www.ucl.ac.uk/~ucahssa/eng/maths/talks.html`.

We shall always work in the unit interval $I = [0,1]$, though most of the arguments are valid in any compact dense-in-itself metric space. For simplicity, by a closed (resp. $\mathcal{F}_\sigma$) set we mean a closed (resp. $\mathcal{F}_\sigma$) subset of I, unless otherwise stated.

In a metric space X, the open ball and the closed ball of centre x and radius r will be denoted by $B_X(x,r)$ and $\bar{B}_X(x,r)$ respectively.

2 Residuality of Families of Closed Sets

In this section we review the residuality of families of closed sets, which is a motivation of this research.

We denote by $\mathcal{K}$ the family of all closed sets. For $K \in \mathcal{K}$ and $r > 0$, we put $K[r] = \bigcup_{x \in K} \bar{B}_I(x,r)$. It is well-known that $\mathcal{K}$ is a compact metric space with respect to the Hausdorff metric d defined by

$$d(K,L) = \inf\bigl(\{\, r > 0 \mid K \subset L[r],\ L \subset K[r] \,\} \cup \{1\}\bigr).$$

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Note that for $K, L \in \mathcal{K}$ and $r \in (0,1)$, we have $d(K,L) \leqq r$ if and only if $K \subset L[r]$ and $L \subset K[r]$, even when either K or L is empty.

Let $\mathcal{F}$ denote the subset of $\mathcal{K}$ consisting of all finite subsets of I; this $\mathcal{F}$ is of great use in studying $\mathcal{K}$.

Proposition 2.1 *The family $\mathcal{F}$ is dense in $\mathcal{K}$.*

Proof. It suffices to prove that for any given $K \in \mathcal{K}$ and $\varepsilon \in (0,1)$ we may find $P \in \mathcal{F}$ with $d(P,K) \leqq \varepsilon$. Since $\bigcup_{x \in K[\varepsilon]} B_I(x,\varepsilon) \supset K$ and K is compact, there exists a finite subset P of $K[\varepsilon]$ for which $\bigcup_{x \in P} B_I(x,\varepsilon) \supset K$. It is easy to see that $d(P,K) \leqq \varepsilon$. ∎

Definition 2.2 We say that a property P is fulfilled by a *typical* closed set if the family of all closed sets satisfying P is residual in $\mathcal{K}$.

Recall that a subset of a topological space is said to be residual if its complement is of first category.

Many properties of a typical closed set have been investigated, and being Lebesgue null is one of them:

Proposition 2.3 *A typical closed set is Lebesgue null.*

Proof. For each positive integer n, we put

$$\mathcal{U}_n = \bigcup_{A \in \mathcal{F} \setminus \{\emptyset\}} B_{\mathcal{K}}\left(A, \frac{1}{n|A|}\right) \cup \{\emptyset\} \subset \mathcal{K},$$

where $|A|$ denotes the cardinality of A. Since the measure of each set in $\mathcal{U}_n$ does not exceed $2/n$, all sets in $\bigcap_{n=1}^{\infty} \mathcal{U}_n$ are null. On the other hand, since Proposition 2.1 shows that $\mathcal{U}_n$ is open and dense in $\mathcal{K}$, the intersection $\bigcap_{n=1}^{\infty} \mathcal{U}_n$ is residual in $\mathcal{K}$. Thus we may conclude that a typical closed set is null. ∎

3 Banach-Mazur Game

This section is devoted to the introduction of the Banach-Mazur game, which is a powerful tool for studying residuality.

Definition 3.1 Let X be a metric space, S a subset of X, and A a dense subset of X. The (X,S,A)-*Banach-Mazur game* is described as follows. Two players, called Player I and Player II, alternately choose a closed ball whose centre belongs to A, with the restriction that they must choose a subset of the ball chosen in the previous turn. Player II will win if the intersection of all the sets chosen by the players is contained in S; otherwise Player I will win.

The following fact enables us to translate residuality in terms of the Banach-Mazur game:

Fact 3.2 *Player II has a winning strategy in the* (X, S, A)*-Banach-Mazur game if and only if* S *is residual in* X*.*

For a slight generalisation of the Banach-Mazur game and the proof of this fact, we refer the reader to Theorem 1 in [1].

In order to show how this game is used to study residuality, we give another proof of Proposition 2.3:

Proof of Proposition 2.3. Let $\mathcal{N}$ denote the family of all null sets in $\mathcal{K}$. Owing to Fact 3.2, we have only to find a winning strategy for Player II in the $(\mathcal{K}, \mathcal{N}, \mathcal{F})$-Banach Mazur game. The move of Player II in his nth turn is determined as follows: if Player I has chosen $\bar{B}_{\mathcal{K}}(A, r)$ in the previous turn, Player II replies $\bar{B}_{\mathcal{K}}(A, r')$, where

$$r' = \begin{cases} \min\{r, 1/n|A|\} & \text{if } A \neq \emptyset; \\ \min\{r, 1/n\} & \text{if } A = \emptyset. \end{cases}$$

Since the radii of the balls chosen by Player II converge to 0 and $\mathcal{K}$ is compact, the intersection of all the balls chosen by the players is a singleton, say $\{K\}$. We can see that $K \in \mathcal{N}$ by the same reasoning as in the first proof of Proposition 2.3. Thus Player II is certain to win according to this strategy, and the proof is complete. ∎

4 Residuality of Families of $\mathcal{F}_\sigma$ Sets

This is the main section of this report, in which we shall state the main result of [2]. Keeping Proposition 2.3 in mind, we may consider it natural to ask whether a typical $\mathcal{F}_\sigma$ set is null. It goes without saying that in order to answer this question we must define what 'typical' means. As we have seen in Section 2, defining 'a typical $\mathcal{F}_\sigma$ set' is equivalent to defining the residuality of families of $\mathcal{F}_\sigma$ sets. We shall give two natural definitions of the residuality.

We adopt the convention that every sequence begins with the term of subscript one and the set $\mathbb{N}$ of all positive integers does not contain zero. The set of all sequences of sets in $\mathcal{K}$ is denoted by $\mathcal{K}^{\mathbb{N}}$ and endowed with the product topology. The closed subset $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ of $\mathcal{K}^{\mathbb{N}}$ is defined as the set of all increasing sequences:

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} = \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; K_1 \subset K_2 \subset \cdots \right\}.$$

Definition 4.1 For a family $\mathcal{F}$ of $\mathcal{F}_\sigma$ sets, we put

$$\mathcal{K}^{\mathbb{N}}_{\mathcal{F}} = \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}.$$

We say that $\mathcal{F}$ is $\mathcal{K}^{\mathbb{N}}$-*residual* if $\mathcal{K}^{\mathbb{N}}_{\mathcal{F}}$ is residual in $\mathcal{K}^{\mathbb{N}}$ and that $\mathcal{F}$ is $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-*residual* if $\mathcal{K}^{\mathbb{N}}_{\mathcal{F}} \cap K^{\mathbb{N}}_{\nearrow}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

The main theorem in [2] asserts that these two notions of residuality agree with each other:

Theorem 4.2 (Main Theorem in [2]) *A family of $\mathcal{F}_\sigma$ sets is $\mathcal{K}^{\mathbb{N}}$-residual if and only if it is $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-residual.*

Moreover the following proposition holds:

Proposition 4.3 *Let $\mathcal{I}$ be a σ-ideal on X. Then $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$ if and only if $\mathcal{I} \cap \mathcal{F}_\sigma$ is $\mathcal{K}^{\mathbb{N}}$-residual.*

This proposition, together with Proposition 2.3, implies that a typical $\mathcal{F}_\sigma$ set is Lebesgue null.

We shall not prove Theorem 4.2 here but shall prove Proposition 4.3. The proof of this proposition we give below is different from that given in [2]; see Remark 4.5 for details. We need a lemma to show the proposition.

Lemma 4.4 *Let X and Y be complete metric spaces, and suppose that $Y \neq \emptyset$. Then a subset A of X is residual if and only if $A \times Y$ is residual in $X \times Y$.*

Proof. We may assume that the distance between two points (a_1, b_1) and (a_2, b_2) in $X \times Y$ is the sum of the distance between a_1 and a_2 in X and the distance between b_1 and b_2 in Y.

The (X, A, X)-Banach Mazur game and the $(X \times Y, A \times Y, X \times Y)$-Banach Mazur game will be called the X-game and the $X \times Y$-game respectively.

We shall prove only that if $A \times Y$ is residual in $X \times Y$ then A is residual in X, since the converse can be shown similarly. Assume that $A \times Y$ is residual in $X \times Y$. Then Player II has a winning strategy in the $X \times Y$-game. The following strategy in the X-game makes Player II win; see Figure 1.

Suppose that Player I has chosen $\bar{B}_X(x_1, r_1)$ in the first turn. Player II takes an element $y_0 \in Y$ and imagines that Player I had chosen $\bar{B}_X\big((x_1, y_0), r_1'\big)$ in the $X \times Y$-game, where r_1' is any positive real number less than $\min\{r_1, 1\}$. Let $\bar{B}_{X \times Y}\big((x_2, y_1), r_2\big)$ denote the ball that Player II would reply according to the strategy in the $X \times Y$-game. The real reply of Player II in the X-game is $\bar{B}_X(x_2, r_2')$, where r_2' is any positive number less than $\min\{r_2, 1/2\}$.

$$
\begin{array}{lcccl}
 & X\text{-game} & & X \times Y\text{-game} & \\
\text{Player I:} & \bar{B}_X(x_1, r_1) & \longrightarrow & \bar{B}_{X\times Y}\big((x_1, y_0), r_1'\big) & \\
\text{Player II:} & \bar{B}_X(x_2, r_2') & \longleftarrow & \bar{B}_{X\times Y}\big((x_2, y_1), r_2\big) & \text{strategy} \\
\text{Player I:} & \bar{B}_X(x_3, r_3) & \longrightarrow & \bar{B}_{X\times Y}\big((x_3, y_1), r_3'\big) & \\
\text{Player II:} & \bar{B}_X(x_4, r_4') & \longleftarrow & \bar{B}_{X\times Y}\big((x_4, y_2), r_4\big) & \text{strategy} \\
\vdots & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & \\
 & x & & (x', y) &
\end{array}
$$

Figure 1: Mechanism of the proof of Lemma 4.4

Suppose that Player I has chosen $\bar{B}_X(x_3, r_3)$ in the next turn. Then Player II imagines that Player I had chosen $\bar{B}_{X\times Y}\big((x_3, y_1), r_3'\big)$ in the $X \times Y$-game, where r_3' is less than $\min\{r_3, r_2 - r_2', 1/3\}$. If $\bar{B}_{X\times Y}\big((x_4, y_2), r_4\big)$ is the ball that Player II would reply according to the strategy, then the real reply is $\bar{B}_X(x_4, r_4')$, where r_4' is less than $\min\{r_4, 1/4\}$.

Player II continues this procedure in deciding each reply. It is easy to verify that $\bar{B}_X(x_{2n}, r_{2n}')$ and $\bar{B}_{X\times Y}\big((x_{2n+1}, y_n), r_{2n+1}'\big)$ are valid replies to $\bar{B}_X(x_{2n-1}, r_{2n-1})$ and to $\bar{B}_{X\times Y}\big((x_{2n}, y_n), r_{2n}\big)$ respectively for all $n \in \mathbb{N}$.

In either game, the intersection of the balls chosen by the players is a singleton. Suppose that $\{x\}$ and $\big\{(x', y)\big\}$ are the intersections. Then we have

$$x = \lim_{n\to\infty} x_n \qquad \text{and} \qquad (x', y) = \lim_{n\to\infty} (x_{2n}, y_n),$$

from which we infer that $x = x'$. It follows from $(x', y) \in A \times Y$ that $x \in A$, and this completes the proof. ∎

Remark 4.5 This proof has the same mechanism as that of the main theorem given in [2], though the latter is technically much more complicated.

In [2], a lemma similar to this one is proved without the Banach-Mazur game; the Kuratowski-Ulam theorem is used instead.

Proof of Proposition 4.3. Since

$$
\begin{aligned}
\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{I} \right\} &= \big\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\big|\; K_n \in \mathcal{I} \text{ for every } n \in \mathbb{N} \big\} \\
&= \bigcap_{n=1}^{\infty} \big(\underbrace{\mathcal{K} \times \cdots \times \mathcal{K}}_{n-1 \text{ times}} \times (\mathcal{I} \cap \mathcal{K}) \times \mathcal{K} \times \mathcal{K} \times \cdots \big),
\end{aligned}
$$

we see that $\mathcal{I} \cap \mathcal{F}_\sigma$ is $\mathcal{K}^{\mathbb{N}}$-residual if and only if $(\mathcal{I} \cap \mathcal{K}) \times \mathcal{K} \times \mathcal{K} \times \cdots$ is residual in $\mathcal{K}^{\mathbb{N}}$. Lemma 4.4 shows that this is equivalent to the condition that $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$. ∎

References

[1] John C. Oxtoby, *The Banach-Mazur Game and Banach Category Theorem*, Contributions to the Theory of Games, vol. 3, Ann. of Math. Stud. **39** (1957), 159–163, Princeton University Press.

[2] Shingo Saito, *Residuality of Families of $\mathcal{F}_\sigma$ Sets*, `math.GN/0506379`, submitted.