Typical $\mathcal{F}_\sigma$ Sets and Typical Continuous Functions

$$\left(\begin{array}{c}\text{Knot Points of}\\ \text{Typical Continuous Functions}\end{array}\right)$$

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Outline

Part I Background

Part II Statement of Main Theorem

Part III Sketch of Proof

Part I
Background

Typical continuous functions

Work in $I := [0, 1]$.

$C(I) := \{f : I \longrightarrow \mathbb{R} \mid f\text{: continuous}\}$
equipped with the supremum norm.

Definition

A typical $f \in C(I)$ satisfies a property $P(f)$
$\overset{\text{def.}}{\Longleftrightarrow}$ $\{f \in C(I) \mid P(f) \text{ holds}\}$ is residual in $C(I)$.

Example

A typical $f \in C(I)$ is nowhere differentiable.

What can we say about Dini derivatives of a typical $f \in C(I)$?

Dini derivatives

Definition (Dini derivatives)

For $f \in C(I)$ and $x \in I$,

$$D^+ f(x) := \limsup_{y \searrow x} \frac{f(y) - f(x)}{y - x},$$

$$D_+ f(x) := \liminf_{y \searrow x} \frac{f(y) - f(x)}{y - x},$$

$$D^- f(x) := \limsup_{y \nearrow x} \frac{f(y) - f(x)}{y - x},$$

$$D_- f(x) := \liminf_{y \nearrow x} \frac{f(y) - f(x)}{y - x}.$$

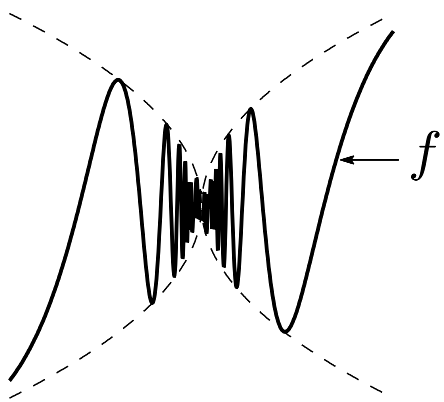
Dini derivatives of a typical $f \in C(I)$

Theorem (Jarník, 1933)

A typical $f \in C(I)$ has the property that

$$D^+ f(x) = D^- f(x) = \infty \text{ and}$$
$$D_+ f(x) = D_- f(x) = -\infty$$

for a.e. $x \in I$.



Such a point x is called a knot point of f.

Knot points of a typical $f \in C(I)$

For $f \in C(I)$,

$$N(f) := \{x \in I \mid x \text{ is NOT a knot point of } f\}.$$

Jarník's theorem asserts that

$N(f)$ is null for a typical $f \in C(I)$.

In what sense of smallness is it true that

$N(f)$ is small for a typical $f \in C(I)$?

Theorem of Preiss and Zajíček

Theorem (Preiss and Zajíček, unpublished)
For a σ-ideal $\mathcal{I}$ on I, T.F.A.E.:

(1) $N(f) \in \mathcal{I}$ for a typical $f \in C(I)$;

(2) $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$.

Here

$$\mathcal{K} := \{K \subset I \mid K \text{ is closed}\}$$

equipped with the Vietoris topology.

(Hausdorff metric)

Problem

Problem

Characterise families $\mathcal{A}$ of subsets of I for which

$$N(f) \in \mathcal{A} \text{ for a typical } f \in C(I).$$

Part II
Statement of Main Theorem

An observation

Problem

Characterise $\mathcal{A} \subset \mathcal{P}(I)$ for which

$$N(f) \in \mathcal{A} \text{ for a typical } f \in C(I).$$

It is easy to see that

$$N(f) \text{ is } \mathcal{F}_\sigma \text{ for all } f \in C(I).$$

Problem

Characterise $\mathcal{F} \subset \mathcal{F}_\sigma$ for which

$$N(f) \in \mathcal{F} \text{ for a typical } f \in C(I).$$

Main Theorem

Main Theorem (S)

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

(1) $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$;

(2) $\mathcal{F}$ is residual in $\mathcal{F}_\sigma$

($F \in \mathcal{F}$ for a typical $F \in \mathcal{F}_\sigma$).

What does residual mean in this context?

Residuality of families of $\mathcal{F}_\sigma$ sets

Proposition (S)

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

(1) $\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \,\middle|\, \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$ is residual in $\mathcal{K}^{\mathbb{N}}$.

(2) $\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \,\middle|\, \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

Here

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} := \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \,\middle|\, K_1 \subset K_2 \subset \cdots \right\}.$$

We say that $\mathcal{F}$ is residual in $\mathcal{F}_\sigma$ if the above conditions hold.

Part III
Sketch of Proof

Statement of main theorem

Main Theorem

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

(1) $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$;

(2) $\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

Proof of Main Thm

Lemma

We may find a 'good' $\mathbb{X} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow} \times C(I)$ s.t.

- $\big((K_n), f\big) \in \mathbb{X}$ implies $\bigcup_{n=1}^{\infty} K_n = N(f)$;
- if $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow}$ is residual, then

  for a typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A} \quad \big((K_n), f\big) \in \mathbb{X}.$$

The proof of this lemma

- is very complicated and
- uses the Banach-Mazur game.

$(2) \Rightarrow (1)$

Suppose

$$\mathcal{A} := \left\{ (K_n) \in \mathcal{K}_{\nearrow}^{\mathbb{N}} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

is residual.

By Lem, for a typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A} \quad \big((K_n), f\big) \in \mathbb{X},$$

$$\therefore N(f) = \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}.$$

Thm (2) $\{(K_n) \in \mathcal{K}_{\nearrow}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ is residual

$\Rightarrow$ (1) $N(f) \in \mathcal{F}$ for a typical f.

Lem $\mathbb{X} \subset \mathcal{K}_{\nearrow}^{\mathbb{N}} \times C(I)$ satisfies

- $\big((K_n), f\big) \in \mathbb{X} \Rightarrow \bigcup_{n=1}^{\infty} K_n = N(f)$;
- $\mathcal{A} \subset \mathcal{K}_{\nearrow}^{\mathbb{N}}$ is residual

$\Rightarrow$ for a typical $f \in C(I)$, $\exists (K_n) \in \mathcal{A}$ $\big((K_n), f\big) \in \mathbb{X}$.

(1) $\Rightarrow$ (2)

Suppose $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$.
Take a dense $\mathcal{G}_\delta$ set $G \subset \{ f \in C(I) \mid N(f) \in \mathcal{F} \}$.
$\mathcal{A} := \{ (K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \exists f \in G \quad ((K_n), f) \in \mathbb{X} \}$.

$(K_n) \in \mathcal{A}$ implies $\bigcup_{n=1}^{\infty} K_n = N(f) \in \mathcal{F}$.
Thus it suffices to show that $\mathcal{A}$ is residual.

Thm (1) $N(f) \in \mathcal{F}$ for a typical f
$\Rightarrow$ (2) $\{ (K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \}$ is residual.

Lem $\mathbb{X} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow} \times C(I)$ satisfies

- $((K_n), f) \in \mathbb{X} \Rightarrow \bigcup_{n=1}^{\infty} K_n = N(f)$;
- $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow}$ is residual

$\Rightarrow$ for a typical $f \in C(I)$, $\exists (K_n) \in \mathcal{A}$ $((K_n), f) \in \mathbb{X}$.

$\mathcal{A}$ turns out to be analytic (since $\mathbb{X}$ is 'good').
$\therefore$ $\mathcal{A}$ has the Baire Property.
$\therefore$ $\mathcal{A}$ is either meagre or residual
(topological 0-1 law).

Suppose $\mathcal{A}$ is meagre.
Then $\mathcal{A}^c$ is residual.

Thm (1) $N(f) \in \mathcal{F}$ for a typical f
$\Rightarrow$ (2) $\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ is residual.

Lem $\mathbb{X} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow} \times C(I)$ satisfies

- $\big((K_n), f\big) \in \mathbb{X} \Rightarrow \bigcup_{n=1}^{\infty} K_n = N(f)$;
- $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow}$ is residual

$\Rightarrow$ for a typical $f \in C(I)$, $\exists (K_n) \in \mathcal{A}$ $\big((K_n), f\big) \in \mathbb{X}$.

Is $\mathcal{A} = \big\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \exists f \in G \quad \big((K_n), f\big) \in \mathbb{X}\big\}$ residual?

By Lem, for a typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A}^c \quad \big((K_n), f\big) \in \mathbb{X}.$$

Thus

$$\exists f \in G \; \exists (K_n) \in \mathcal{A}^c \quad \big((K_n), f\big) \in \mathbb{X}.$$

This contradicts the definition of $\mathcal{A}$. ∎

Thm (1) $N(f) \in \mathcal{F}$ for a typical f
$\Rightarrow$ (2) $\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ is residual.

Lem $\mathbb{X} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow} \times C(I)$ satisfies

- $\big((K_n), f\big) \in \mathbb{X} \Rightarrow \bigcup_{n=1}^{\infty} K_n = N(f)$;
- $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}_{\nearrow}$ is residual

$\Rightarrow$ for a typical $f \in C(I)$, $\exists (K_n) \in \mathcal{A}$ $\big((K_n), f\big) \in \mathbb{X}$.

$\mathcal{A}^c = \big\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \;\big|\; \forall f \in G \; \big((K_n), f\big) \notin \mathbb{X}\big\}$ is assumed to be residual.