Knot Points of Typical Continuous Functions

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Definition

A typical (= generic) $f \in C(I)$ has property P
(written $\forall^* f \in C(I) \;\; P$)
$\stackrel{\text{def}}{\iff} \{f \in C(I) \mid f \text{ has property } P\}$ is residual.

residual = comeagre = complement is meagre

meagre = first category

= countable union of nowhere dense sets

Properties of typical continuous functions

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What about

$$\overline{D}f(x) = \limsup_{y \to x} \frac{f(y) - f(x)}{y - x},$$

$$\underline{D}f(x) = \liminf_{y \to x} \frac{f(y) - f(x)}{y - x}?$$

$\overline{D}$ and $\underline{D}$ of typical continuous functions

Theorem (Jarník, 1933)

$\forall^* f \in C(I)\ \forall x \in I\ \ \overline{D}f(x) = \infty,\ \underline{D}f(x) = -\infty.$

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What about Dini derivatives

$$D^+ f(x) = \limsup_{y \downarrow x} \frac{f(y)-f(x)}{y-x},$$

$$D_+ f(x) = \liminf_{y \downarrow x} \frac{f(y)-f(x)}{y-x},$$

$$D^- f(x) = \limsup_{y \uparrow x} \frac{f(y)-f(x)}{y-x},$$

$$D_- f(x) = \liminf_{y \uparrow x} \frac{f(y)-f(x)}{y-x}?$$

Dini derivatives of typical functions

Is it true that
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No!
There is no such f.
For, if f attains its maximum at $a \in I$, then
$D^+ f(a) \le 0$ and $D_- f(a) \ge 0$.

Knot points

Definition

A point $x \in I$ is a knot point of $f \in C(I)$ if

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$\nexists f \in C(I)\ \forall x \in I \quad x$ is a knot point of f.
Nevertheless, it turns out that
$\forall^* f \in C(I)$ 'most' points are knot points of f.

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In other words, setting

$$N(f) = \{x \in I \mid x \text{ is NOT a knot point of } f\},$$

we have

$$\forall^* f \in C(I) \quad N(f) \text{ is Lebesgue null.}$$

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How small is $N(f)$ for a typical $f \in C(I)$?

Theorem of Preiss and Zajíček

Theorem (Preiss and Zajíček, unpublished)

For a σ-ideal $\mathcal{I}$ on I, T.F.A.E.:

(1) $\forall^* f \in C(I) \;\; N(f) \in \mathcal{I}$;

(2) $\mathcal{I} \cap \mathcal{K}$ *is residual in* $\mathcal{K}$ *(i.e.* $\forall^* K \in \mathcal{K} \;\; K \in \mathcal{I}$*).*

Here $\mathcal{K} = \{K \subset I \mid K \text{ is closed } (= \text{compact})\}$ with the Hausdorff metric (Vietoris topology).

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Example

$\forall^* f \in C(I) \;\; N(f)$ is meagre, $\dim_{\mathrm{H}} N(f) = 0$, etc.

Generalisation

Problem

Characterise $\mathcal{S} \subset \mathcal{P}(I)$ *s.t.* $\forall^* f \in C(I) \quad N(f) \in \mathcal{S}$.

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Problem

Characterise $\mathcal{F} \subset \mathcal{F}_\sigma$ *s.t.* $\forall^* f \in C(I) \;\; N(f) \in \mathcal{F}$.

Main Theorem

Theorem (Preiss and S.)

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

(1) $\forall^* f \in C(I) \quad N(f) \in \mathcal{F}$;

(2) $\forall^* (K_n) \in \mathcal{K}^{\mathbb{N}} \quad \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$.

(2) means that the inverse image of $\mathcal{F}$ under the surjection

$$\mathcal{K}^{\mathbb{N}} \longrightarrow \mathcal{F}_\sigma; (K_n) \longmapsto \bigcup_{n=1}^{\infty} K_n$$

is residual.

Attempt to prove the main theorem

$$\underline{(2)\ \forall^*(K_n)\quad \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \implies (1)\ \forall^* f\quad N(f) \in \mathcal{F}}$$

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Suffices to show:
$\{f \in C(I) \mid \exists (K_n) \in \mathcal{A} \; N(f) = \bigcup_{n=1}^{\infty} K_n\}$ is residual.

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This is possible by constructing a very complicated winning strategy in a Banach-Mazur game.
($\exists$ winning strategy in BM-game $\iff$ residuality)

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This is possible by constructing a very complicated winning strategy in a Banach-Mazur game.
($\exists$ winning strategy in BM-game $\iff$ residuality)
But the converse appears to be difficult to prove by the same method.

Sophistication of this method

Lemma

$\exists \mathbb{X} \subset \mathcal{K}^{\mathbb{N}} \times C(I)$ *s.t.*

(A) $\big((K_n), f\big) \in \mathbb{X} \implies \bigcup_{n=1}^{\infty} K_n = N(f)$;

(B) $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}$ *is residual*

$\implies \forall^* f \in C(I)\ \exists (K_n) \in \mathcal{A}\ \ \big((K_n), f\big) \in \mathbb{X}$;

(C) $\mathbb{X}$ *is analytic* $(= \mathbf{\Sigma}_1^1)$ *in* $\mathcal{K}^{\mathbb{N}} \times C(I)$;

(D) *one more condition (specified later).*

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(D) *one more condition (specified later).*

We define $\mathbb{X}$ in a (complicated but) explicit way.
Use the BM-game to show (B) (hardest part in the whole proof).

$(2) \implies (1)$, assuming Lemma

(2) $\forall^* (K_n) \quad \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \implies$ (1) $\forall^* f \quad N(f) \in \mathcal{F}$

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$\therefore \forall^* f \in C(I)\, \exists (K_n) \in \mathcal{A}\ \bigcup_{n=1}^{\infty} K_n = N(f)$ by (A).
$\therefore \forall^* f \in C(I) \quad N(f) \in \mathcal{F}$ by def of $\mathcal{A}$. □

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Take a dense G_δ set $G \subset \{f \in C(I) \mid N(f) \in \mathcal{F}\}$.

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Suffices to show that $\mathcal{A}$ is residual.

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$\mathcal{A}$ is nonmeagre.

For, if $\mathcal{A}$ is meagre, then applying (B) to $\mathcal{A}^c$ gives

$$\forall^* f \in C(I) \; \exists (K_n) \in \mathcal{A}^c \;\; \big((K_n), f\big) \in \mathbb{X}.$$

So $\exists f \in G \; \exists (K_n) \in \mathcal{A}^c \;\; \big((K_n), f\big) \in \mathbb{X}$,

contradicting the def of $\mathcal{A}$.

Topological 0-1 law, finishing the proof

Apply a topological 0-1 law to

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- $\mathcal{A} = \bigcup_{f \in G}\{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \big((K_n), f\big) \in \mathbb{X}\}$ is invariant under finite permutations because (D) $\forall f \in C(I) \;\; \{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \big((K_n), f\big) \in \mathbb{X}\}$ is invariant under finite permutations.

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It follows that $\mathcal{A}$ is either meagre or residual.

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Since $\mathcal{A}$ is nonmeagre, $\mathcal{A}$ must be residual. $\square$