Residuality of Families of $\mathcal{F}_\sigma$ Sets

Shingo SAITO
(斎藤新悟)
University College London
http://www.ucl.ac.uk/~ucahssa/

1 Residuality of Families of Closed Sets

Work in $I := [0, 1]$.

$\mathcal{K} := \{K \subset I \mid K\text{: closed}\}$.

d: Hausdorff metric on $\mathcal{K}$.

($d(K, \emptyset) := 1$ for nonempty $K \in \mathcal{K}$.)

d defines residuality on $\mathcal{K}$.

e.g.

$\{K \in \mathcal{K} \mid K\text{: null}\}$: residual in $\mathcal{K}$.

We say typical closed sets are null.

2 Residuality of Families of $\mathcal{F}_\sigma$ Sets

2.1 How about giving $\mathcal{F}_\sigma$ a topology?

Give $\mathcal{F}_\sigma$ a topology via the surjection

$$\begin{array}{ccc} \mathcal{K}^{\mathbb{N}} & \longrightarrow & \mathcal{F}_\sigma \\ \Cup & & \Cup \\ (K_n) & \longmapsto & \bigcup_{n=1}^{\infty} K_n \end{array}$$

It turns out that

$$\mathcal{F} \subset \mathcal{F}_\sigma\text{: residual} \iff \{F \in \mathcal{F}_\sigma \mid F\text{: dense}\} \subset \mathcal{F}$$

This residuality is not useful.

2.2 Another approach

Def

$\mathcal{F} \subset \mathcal{F}_\sigma$ is residual if

$$\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

is residual in $\mathcal{K}^{\mathbb{N}}$.

Another natural surjection:

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} := \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; K_1 \subset K_2 \subset \cdots \right\}$$

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} \longrightarrow \mathcal{F}_\sigma; \;\; (K_n) \longmapsto \bigcup_{n=1}^{\infty} K_n$$

Def

$\mathcal{F} \subset \mathcal{F}_\sigma$ is $\nearrow$-residual if

$$\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

Main Thm

$\mathcal{F} \subset \mathcal{F}_\sigma$ is residual iff it is $\nearrow$-residual.

2.3 Evidences that this is *the* definition

Thm

For a σ-ideal $\mathcal{I}$ on I, T.F.A.E.:

(1) $\mathcal{I} \cap \mathcal{F}_\sigma$ is residual in $\mathcal{F}_\sigma$;

(2) $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$.

Cor

Typical $\mathcal{F}_\sigma$ sets are null.

Relations with Functions

$C(I) := \{f\colon I \longrightarrow \mathbb{R} \mid f\colon \text{continuous}\}$

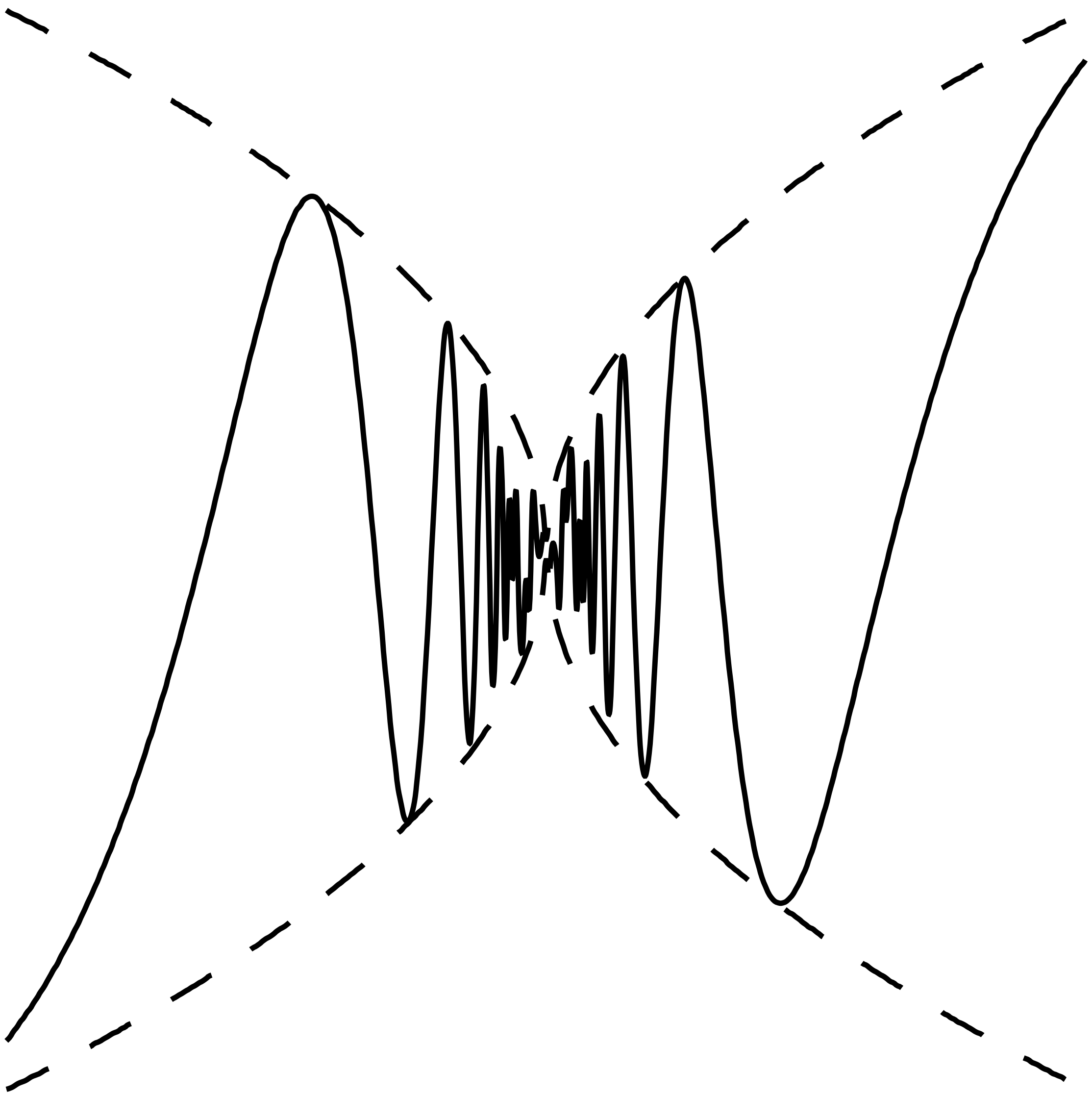
with the sup norm $\|\cdot\|$

Def

$x \in I$ is a knot point of $f \in C(I)$ if

$$D^+f(x) := \limsup_{y \searrow x} \frac{f(y) - f(x)}{y - x} = \infty,$$

$$D_+f(x) = -\infty,\ D^-f(x) = \infty,\ D_-f(x) = -\infty.$$



For $f \in C(I)$,

$$N(f) := \{x \in I \mid x \text{ is not a knot point of } f\} \in \mathcal{F}_\sigma.$$

Thm (Preiss and Zajíček (unpublished))

For a σ-ideal $\mathcal{I}$ on I, T.F.A.E.:

(1) $N(f) \in \mathcal{I}$ for typical $f \in C(I)$;

(2) $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$.

Thm

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

(1) $N(f) \in \mathcal{F}$ for typical $f \in C(I)$;

(2) $\mathcal{F}$ is residual in $\mathcal{F}_\sigma$.

3 Proof of Main Thm

Main Thm

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

(1) $\left\{(K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\right\}$: residual in $\mathcal{K}^{\mathbb{N}}$;

(2) $\left\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\right\}$: residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

3.1 Banach-Mazur game

X: metric space, $S \subset X$.

Players ① and ② alternately choose a ball.

They must choose a subset of the ball chosen in the previous turn.

$$\begin{array}{ccccccccc} B_1 & \supset & B_2 & \supset & B_3 & \supset & B_4 & \supset & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ ① & & ② & & ① & & ② & & \end{array}$$

Player ② wins iff $\bigcap_{n=1}^{\infty} B_n \subset S$.

Fact

Player ② has a winning strategy iff S is residual.

3.2 Plan for the proof

We look at

$$\nearrow\text{-residual} \implies \text{residual.}$$

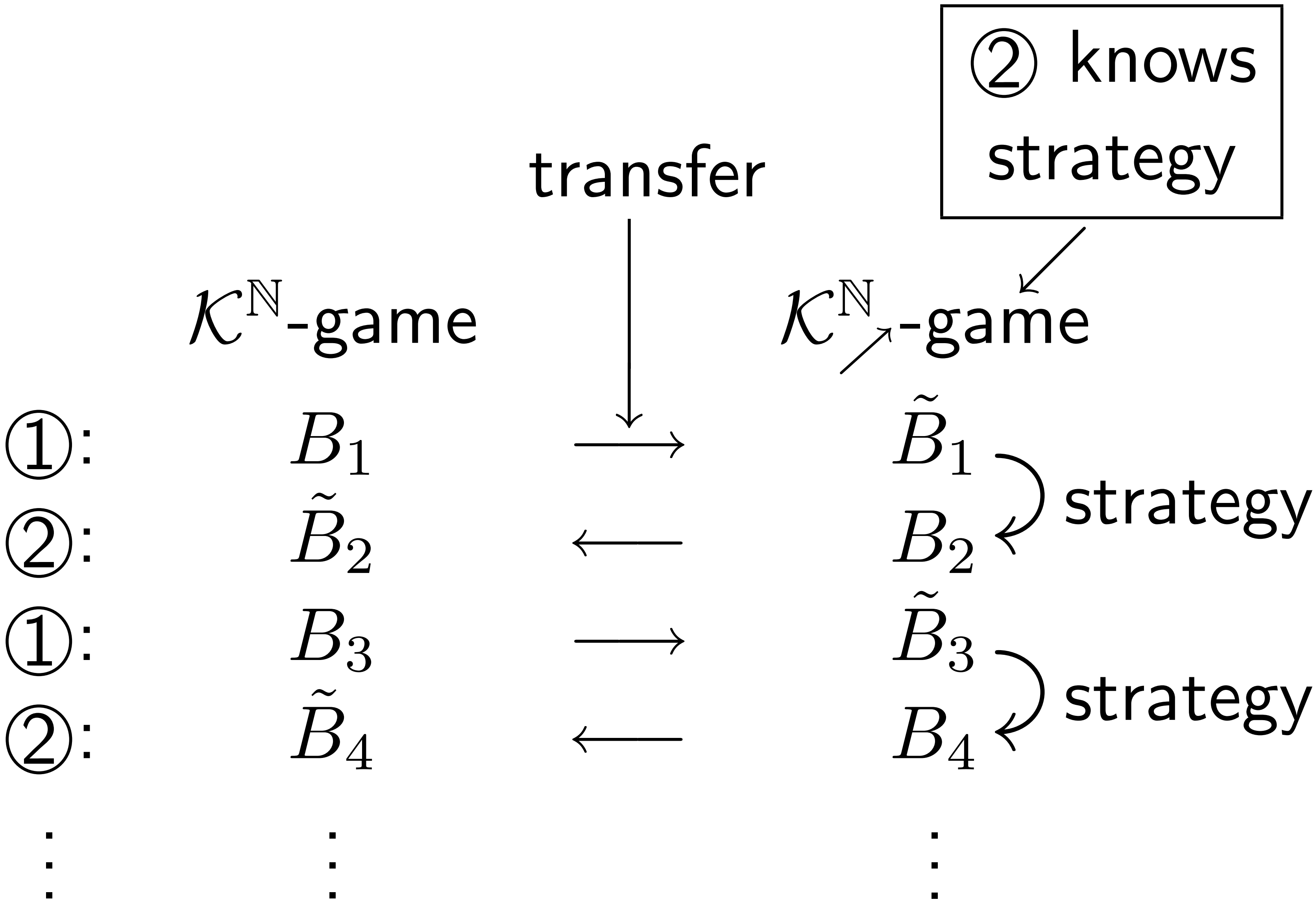
$\mathcal{F} \subset \mathcal{F}_\sigma$: $\nearrow$-residual.

$\mathcal{K}^{\mathbb{N}}_{\nearrow}$-game

- $X = \mathcal{K}^{\mathbb{N}}_{\nearrow}$.
- $S = \left\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \,\middle|\, \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\right\}$.
- Player ② knows a winning strategy.

$\mathcal{K}^{\mathbb{N}}$-game

- $X = \mathcal{K}^{\mathbb{N}}$.
- $S = \left\{(K_n) \in \mathcal{K}^{\mathbb{N}} \,\middle|\, \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\right\}$.
- Player ② is looking for a winning strategy.


② knows strategy
transfer
$\mathcal{K}^{\mathbb{N}}$-game
$\mathcal{K}^{\mathbb{N}}_{\nearrow}$-game
①: B_1 ⟶ $\tilde{B}_1$
②: $\tilde{B}_2$ ⟵ B_2
①: B_3 ⟶ $\tilde{B}_3$
②: $\tilde{B}_4$ ⟵ B_4
strategy
strategy

3.3 Transfers

What matters is only the centres (not the radii).

We may assume that the centres are

sequences of disjoint finite sets in $\mathcal{K}^{\mathbb{N}}$-game,

sequences of finite sets in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-game.

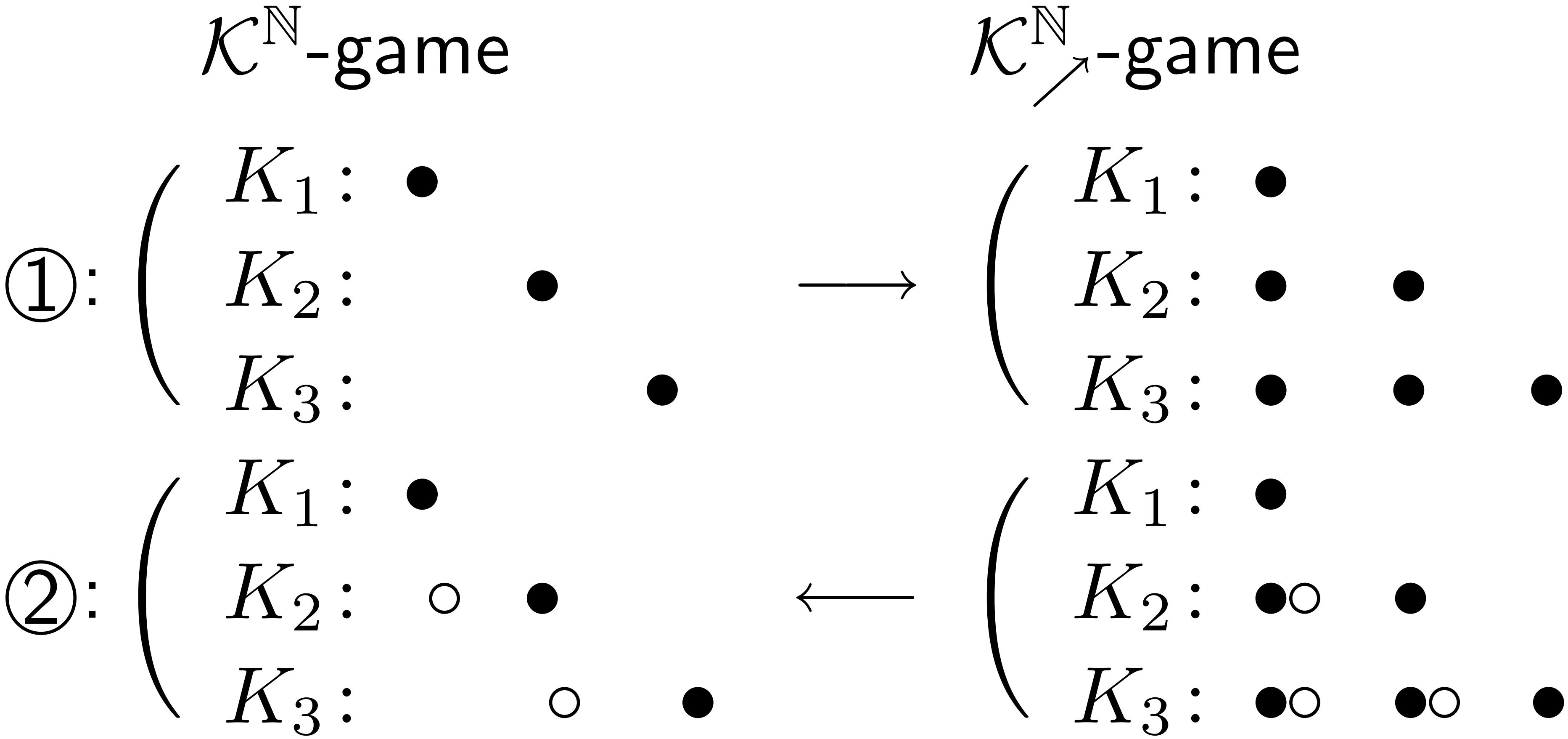
How about the maps

$$\mathcal{K}^{\mathbb{N}} \longrightarrow \mathcal{K}^{\mathbb{N}}_{\nearrow};\ (K_n) \longmapsto (K_1, K_1 \cup K_2, \ldots),$$

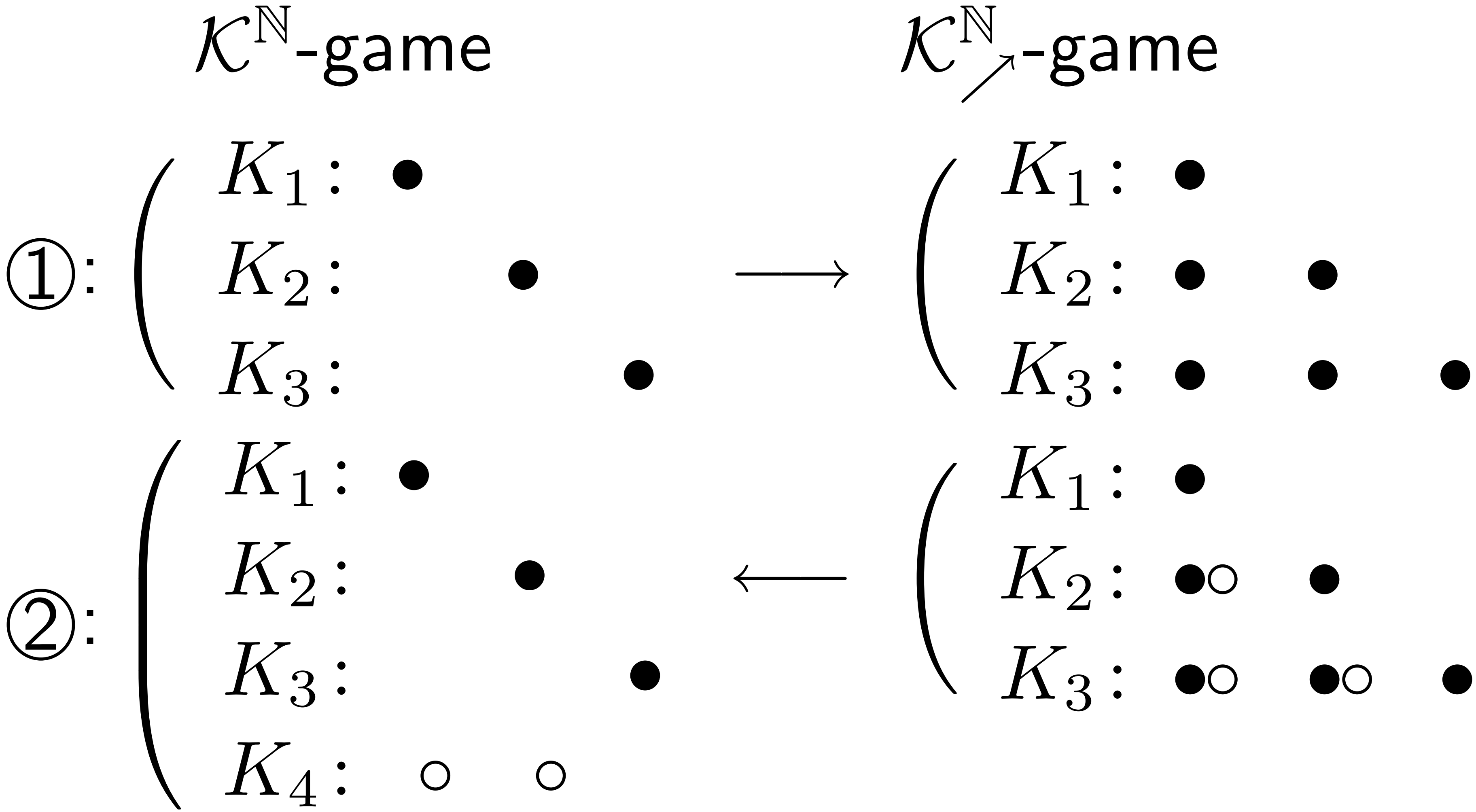
$$\mathcal{K}^{\mathbb{N}}_{\nearrow} \longrightarrow \mathcal{K}^{\mathbb{N}};\ (K_n) \longmapsto (K_1, K_2 \setminus K_1, \ldots)?$$

They don't work.

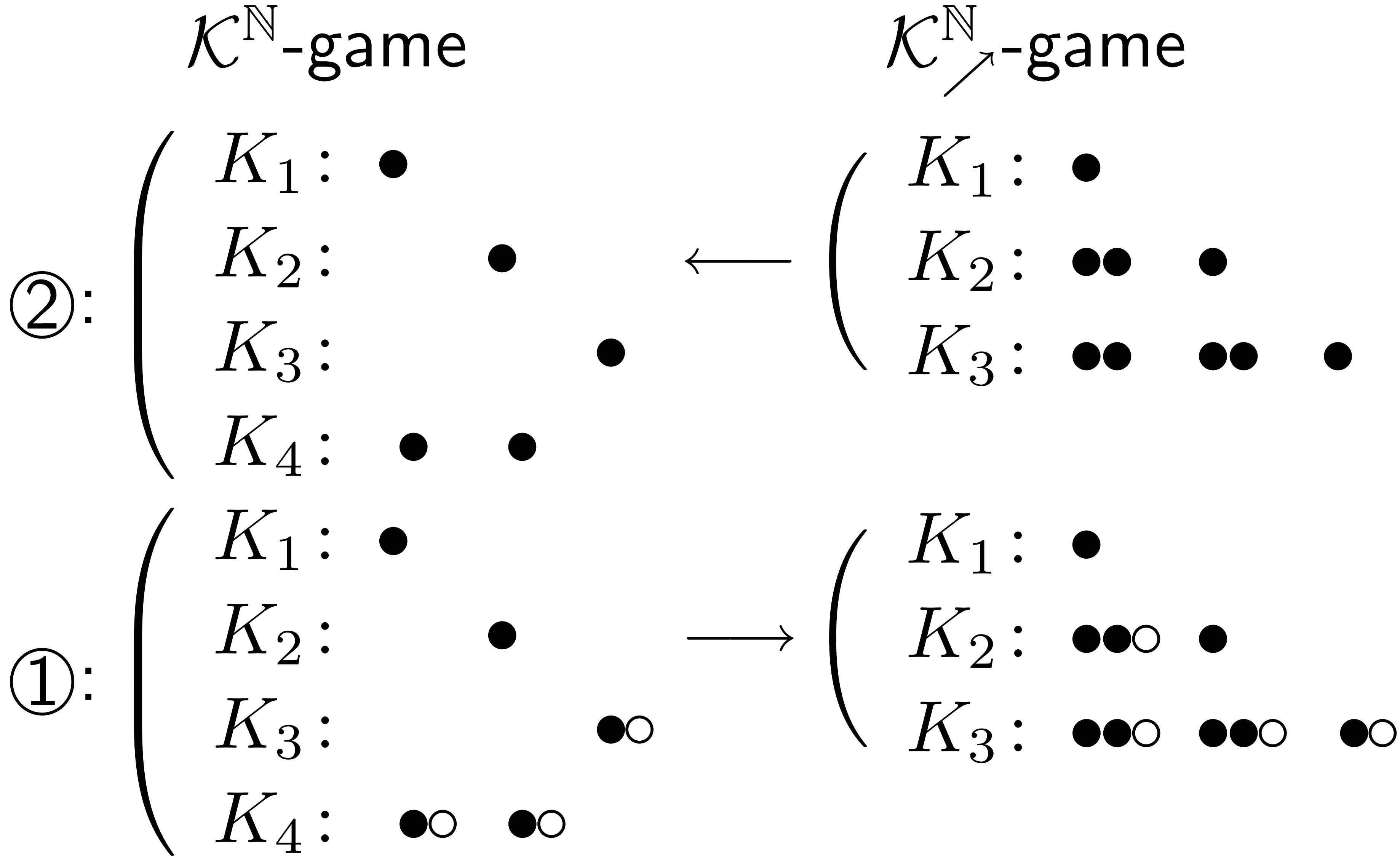
Why don't they work?



Throw these points ○ away:



Next step:



$\mathcal{K}^{\mathbb{N}}$-game
$\mathcal{K}^{\mathbb{N}}_{\nearrow}$-game
②:
K_1:
K_2:
K_3:
K_4:
①:
K_1:
K_2:
K_3:
K_4:
K_1:
K_2:
K_3:
K_1:
K_2:
K_3:

3.4 What remains to be proved

Suppose

$$\text{(intersection of balls in } \mathcal{K}^{\mathbb{N}}\text{-game)} = \{(P_n)\},$$

$$\text{(intersection of balls in } \mathcal{K}^{\mathbb{N}}_{\nearrow}\text{-game)} = \{(Q_n)\}.$$

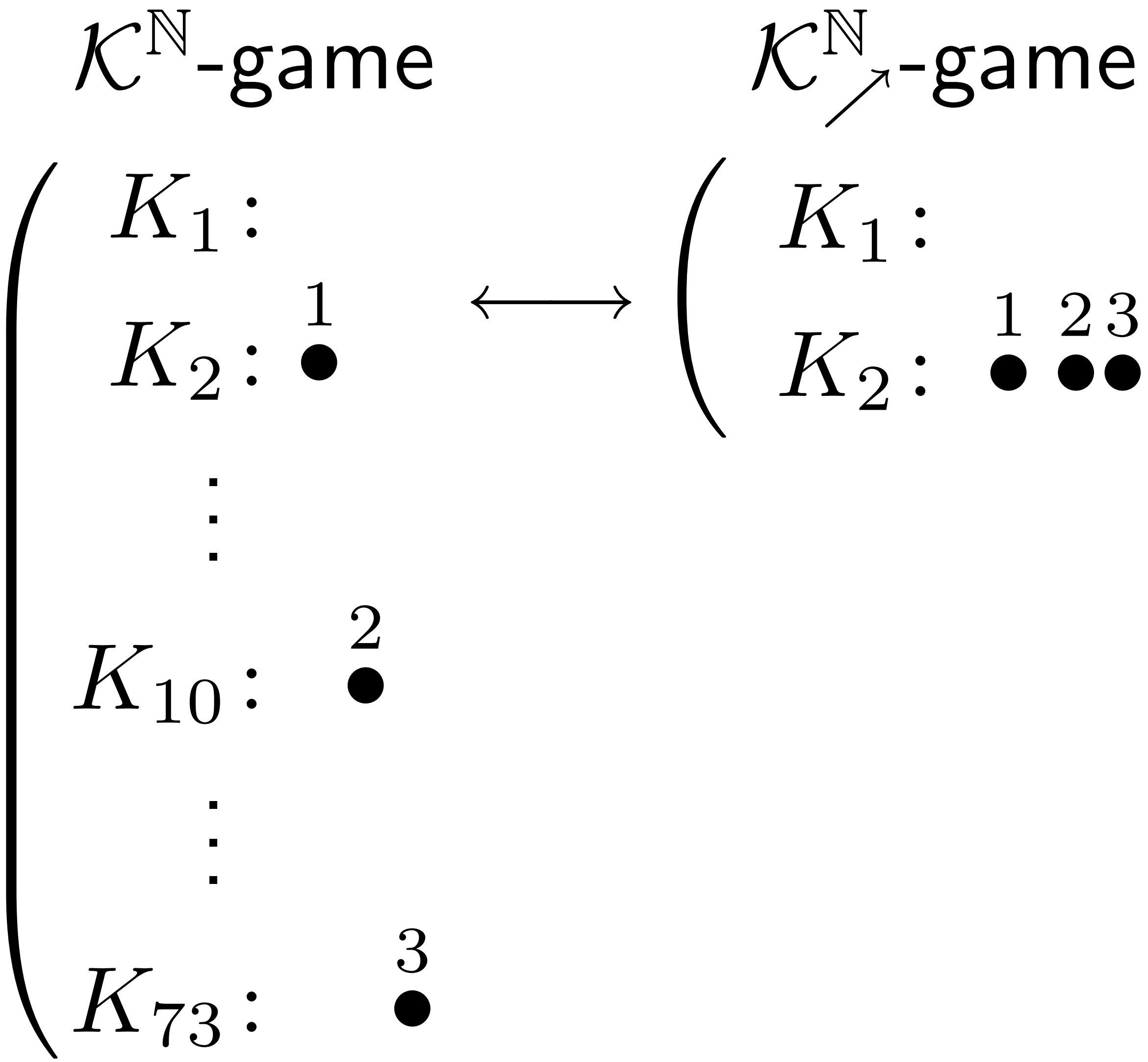
(P_n), (Q_n): limit of centres.

We know $\bigcup_{n=1}^{\infty} Q_n \in \mathcal{F}$.

We want $\bigcup_{n=1}^{\infty} P_n \in \mathcal{F}$,

so shall prove $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$.

Problem



The limit must belong to P_1.