
Residuality of Families of \mathcal{F}_σ Sets

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1 Residuality of Families of Closed Sets

Work in $I := [0, 1]$.

$\mathcal{K} := \{K \subset I \mid K: \text{closed}\}$.

d : Hausdorff metric on \mathcal{K} .

($d(K, \emptyset) := 1$ for nonempty $K \in \mathcal{K}$.)

d defines residuality on \mathcal{K} .

e.g.

$\{K \in \mathcal{K} \mid K: \text{null}\}$: residual in \mathcal{K} .

We say **typical** closed sets are null.

2 Residuality of Families of \mathcal{F}_σ Sets

2.1 How about giving \mathcal{F}_σ a topology?

Give \mathcal{F}_σ a topology via the surjection

$$\begin{array}{ccc} \mathcal{K}^{\mathbb{N}} & \longrightarrow & \mathcal{F}_\sigma \\ \Downarrow & & \Downarrow \\ (K_n) & \longmapsto & \bigcup_{n=1}^{\infty} K_n \end{array}$$

It turns out that

$$\mathcal{F} \subset \mathcal{F}_\sigma: \text{residual} \iff \{F \in \mathcal{F}_\sigma \mid F: \text{dense}\} \subset \mathcal{F}$$

This residuality is not useful.

2.2 Another approach

Def

$\mathcal{F} \subset \mathcal{F}_\sigma$ is **residual** if

$$\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

is residual in $\mathcal{K}^{\mathbb{N}}$.

Another natural surjection:

$$\mathcal{K}_{\nearrow}^{\mathbb{N}} := \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \mid K_1 \subset K_2 \subset \cdots \right\}$$

$$\mathcal{K}_{\nearrow}^{\mathbb{N}} \longrightarrow \mathcal{F}_{\sigma}; \quad (K_n) \longmapsto \bigcup_{n=1}^{\infty} K_n$$

Def

$\mathcal{F} \subset \mathcal{F}_{\sigma}$ is \nearrow -residual if

$$\left\{ (K_n) \in \mathcal{K}_{\nearrow}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

is residual in $\mathcal{K}_{\nearrow}^{\mathbb{N}}$.

Main Thm

$\mathcal{F} \subset \mathcal{F}_\sigma$ is residual iff it is \nearrow -residual.

2.3 Evidences that this is *the* definition

Thm

For a σ -ideal \mathcal{I} on I , T.F.A.E.:

- (1) $\mathcal{I} \cap \mathcal{F}_\sigma$ is residual in \mathcal{F}_σ ;
- (2) $\mathcal{I} \cap \mathcal{K}$ is residual in \mathcal{K} .

Cor

Typical \mathcal{F}_σ sets are null.

Relations with Functions

$C(I) := \{f : I \longrightarrow \mathbb{R} \mid f: \text{continuous}\}$

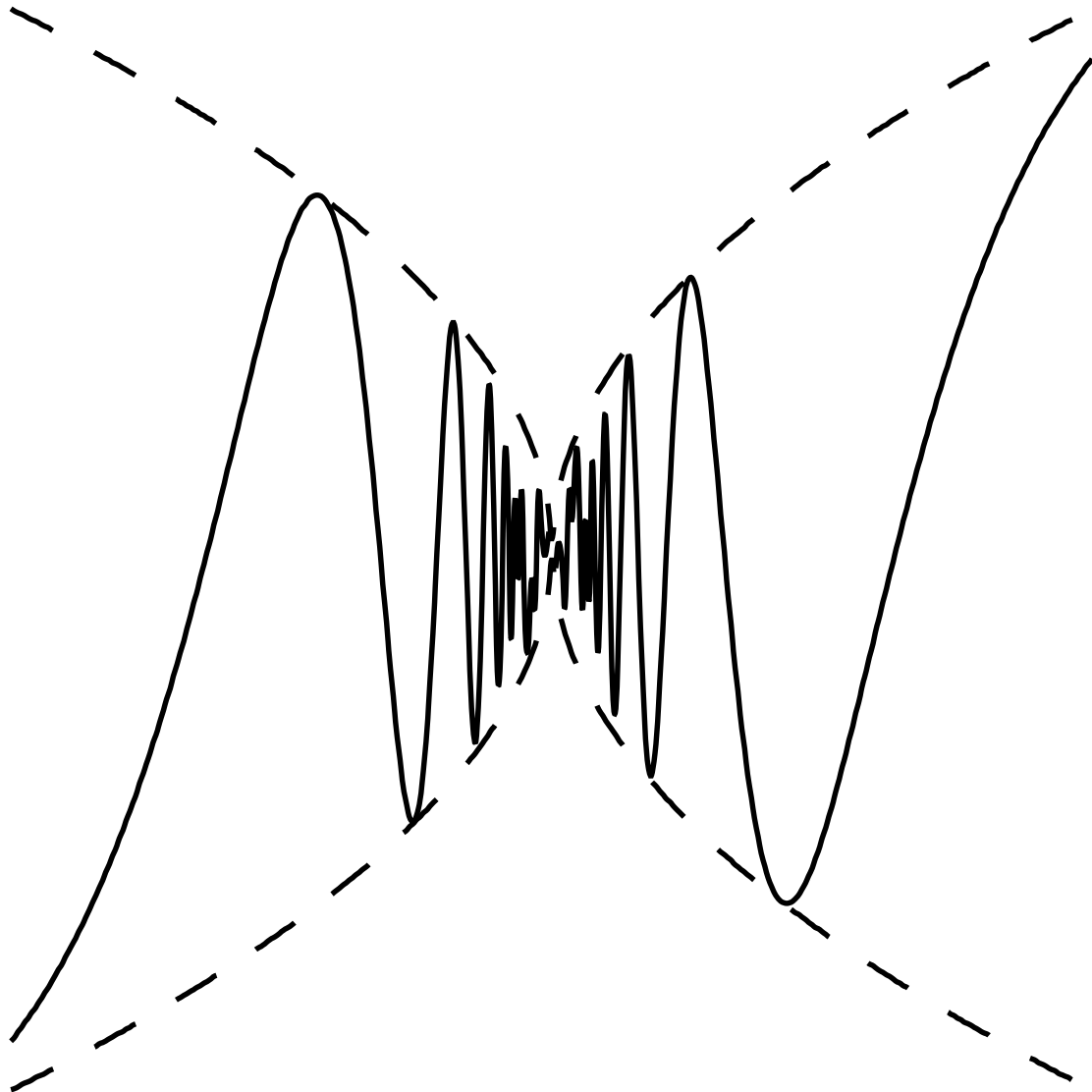
with the sup norm $\|\cdot\|$

Def

$x \in I$ is a **knot point** of $f \in C(I)$ if

$$D^+ f(x) := \limsup_{y \searrow x} \frac{f(y) - f(x)}{y - x} = \infty,$$

$$D_+ f(x) = -\infty, \quad D^- f(x) = \infty, \quad D_- f(x) = -\infty.$$



For $f \in C(I)$,

$$N(f) := \{x \in I \mid x \text{ is not a knot point of } f\} \\ \in \mathcal{F}_\sigma.$$

Thm (Preiss and Zajíček (unpublished))

For a σ -ideal \mathcal{I} on I , T.F.A.E.:

- (1) $N(f) \in \mathcal{I}$ for typical $f \in C(I)$;
- (2) $\mathcal{I} \cap \mathcal{K}$ is residual in \mathcal{K} .

Thm

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

- (1) $N(f) \in \mathcal{F}$ for typical $f \in C(I)$;
- (2) \mathcal{F} is residual in \mathcal{F}_σ .

3 Proof of Main Thm

Main Thm

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

- (1) $\{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$: residual in $\mathcal{K}^{\mathbb{N}}$;
- (2) $\{(K_n) \in \mathcal{K}_{\nearrow}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$: residual in $\mathcal{K}_{\nearrow}^{\mathbb{N}}$.

3.1 Banach-Mazur game

X : metric space, $S \subset X$.

Players ① and ② alternately choose a ball.

They must choose a subset of the ball chosen in the previous turn.

$$\begin{array}{ccccccc} B_1 & \supset & B_2 & \supset & B_3 & \supset & B_4 & \supset & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \textcircled{1} & & \textcircled{2} & & \textcircled{1} & & \textcircled{2} & & \end{array}$$

Player ② wins iff $\bigcap_{n=1}^{\infty} B_n \subset S$.

Fact

Player ② has a winning strategy iff S is residual.

3.2 Plan for the proof

We look at

\nearrow -residual \implies residual.

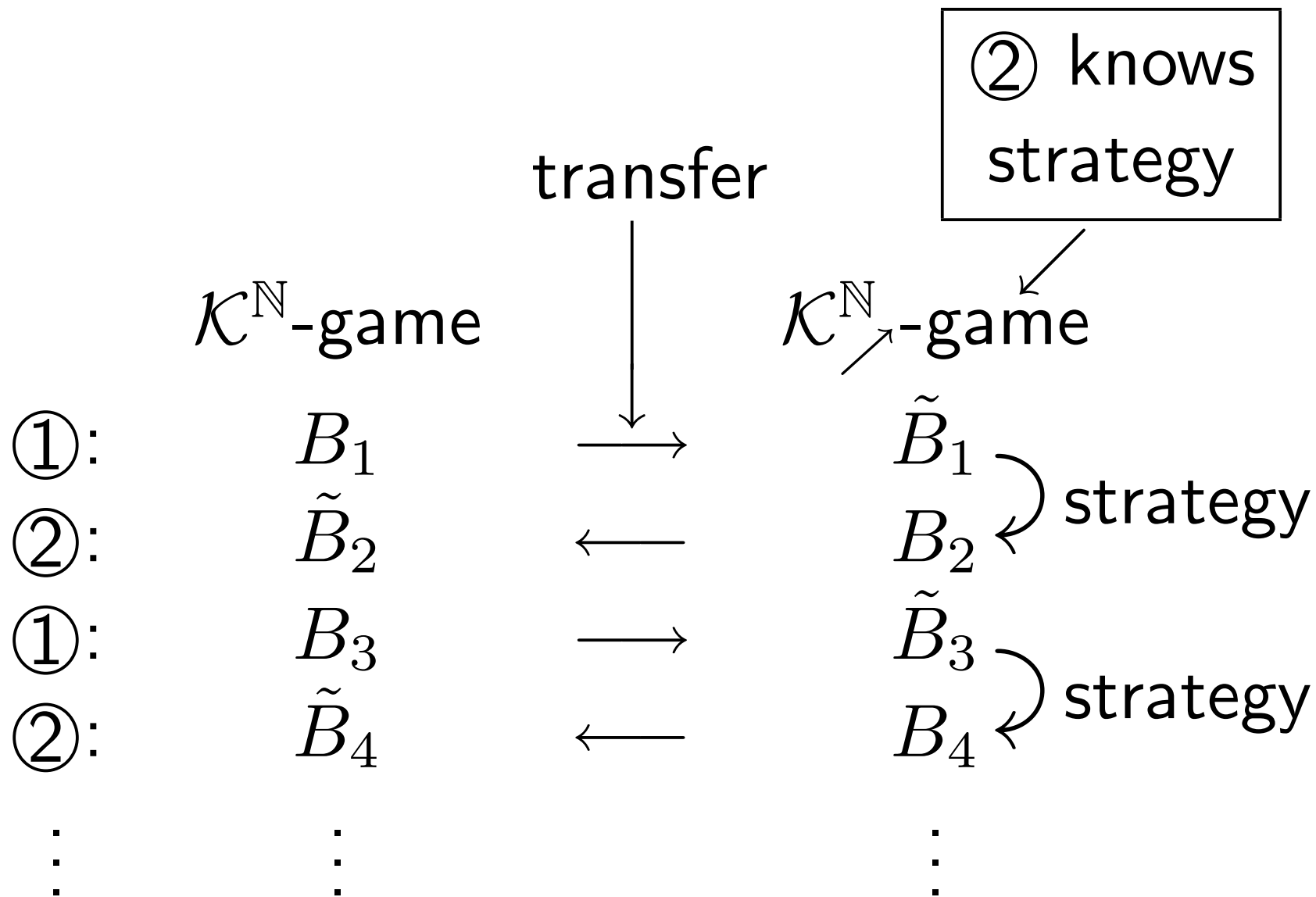
$\mathcal{F} \subset \mathcal{F}_\sigma$: \nearrow -residual.

$\mathcal{K}_{\nearrow}^{\mathbb{N}}$ -game

- $X = \mathcal{K}_{\nearrow}^{\mathbb{N}}$.
- $S = \left\{ (K_n) \in \mathcal{K}_{\nearrow}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$.
- Player ② knows a winning strategy.

$\mathcal{K}^{\mathbb{N}}$ -game

- $X = \mathcal{K}^{\mathbb{N}}$.
- $S = \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$.
- Player ② is looking for a winning strategy.



3.3 Transfers

What matters is only the centres (not the radii).

We may assume that the centres are

sequences of disjoint finite sets in $\mathcal{K}^{\mathbb{N}}$ -game,

sequences of finite sets in $\mathcal{K}_{\nearrow}^{\mathbb{N}}$ -game.

How about the maps

$$\mathcal{K}^{\mathbb{N}} \longrightarrow \mathcal{K}_{\nearrow}^{\mathbb{N}}; (K_n) \longmapsto (K_1, K_1 \cup K_2, \dots),$$

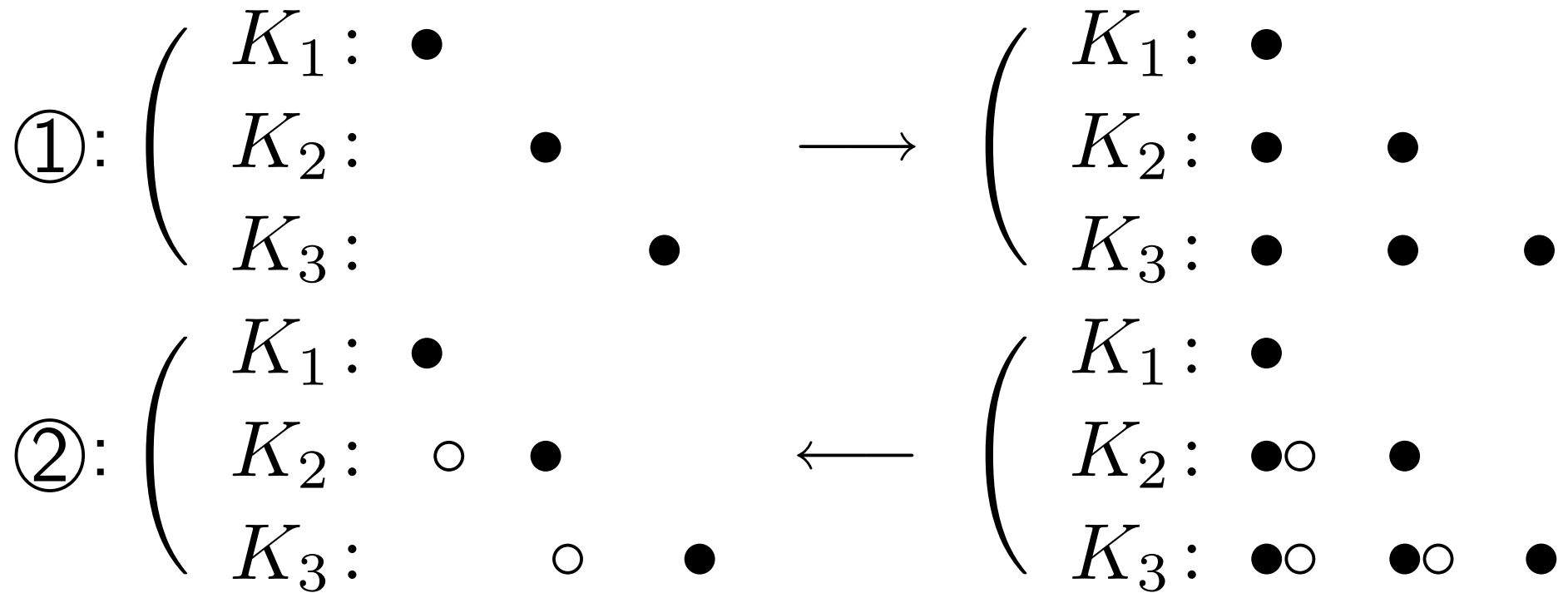
$$\mathcal{K}_{\nearrow}^{\mathbb{N}} \longrightarrow \mathcal{K}^{\mathbb{N}}; (K_n) \longmapsto (K_1, K_2 \setminus K_1, \dots)?$$

They don't work.

Why don't they work?

$\mathcal{K}^{\mathbb{N}}$ -game

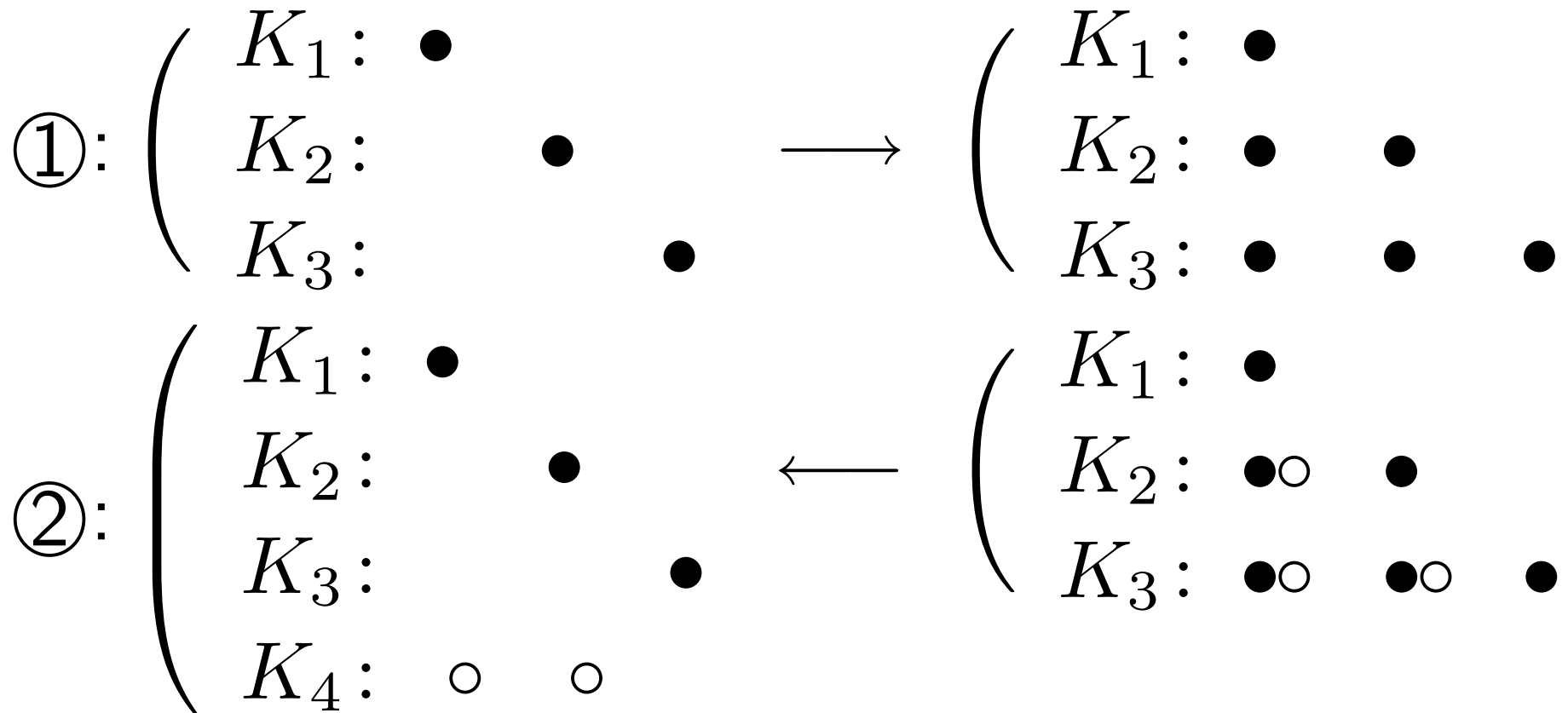
$\mathcal{K}_{\nearrow}^{\mathbb{N}}$ -game



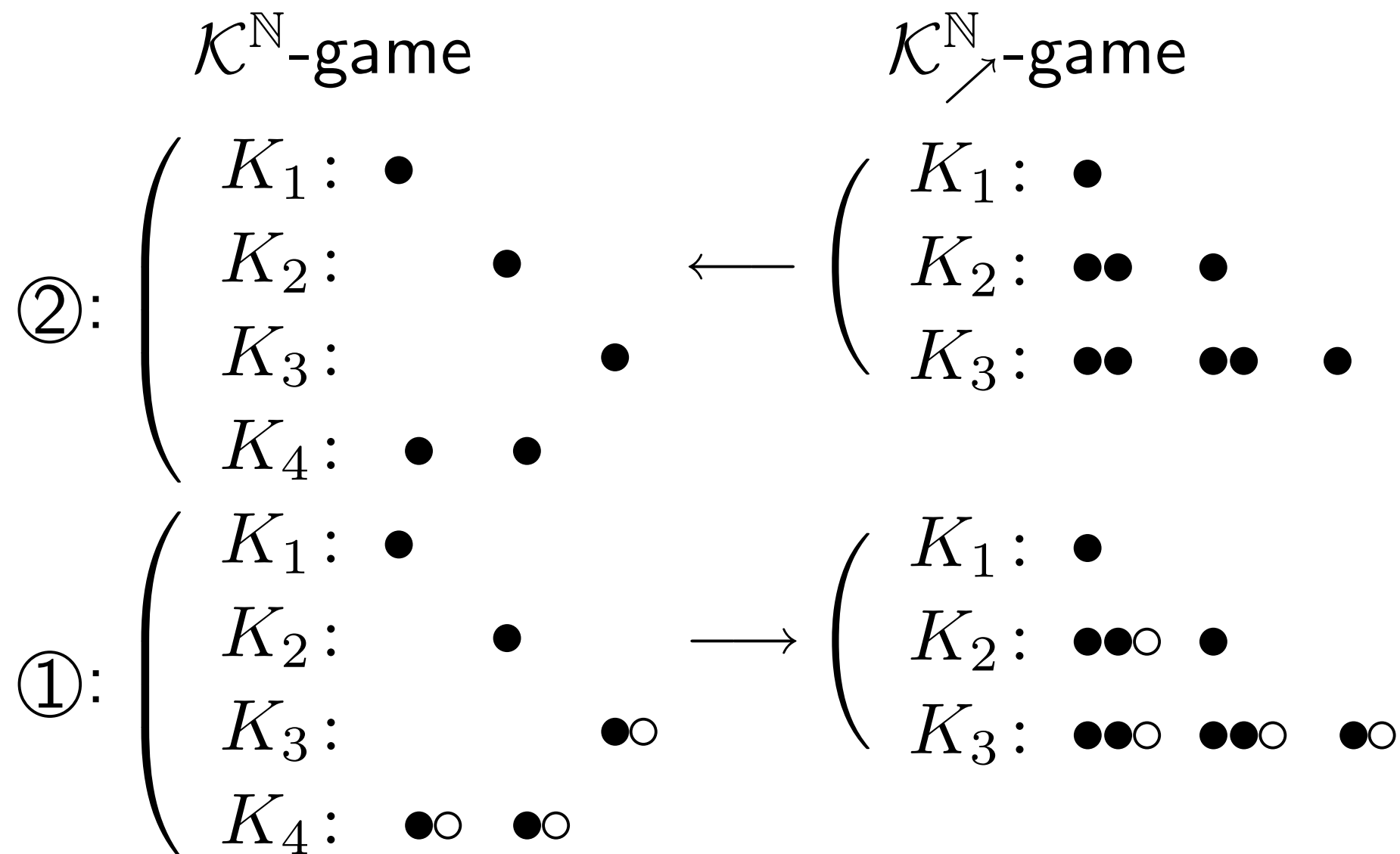
Throw these points \circ away:

$\mathcal{K}^{\mathbb{N}}$ -game

$\mathcal{K}_{\nearrow}^{\mathbb{N}}$ -game



Next step:



3.4 What remains to be proved

Suppose

(intersection of balls in $\mathcal{K}^{\mathbb{N}}$ -game) = $\{(P_n)\}$,

(intersection of balls in $\mathcal{K}_{\nearrow}^{\mathbb{N}}$ -game) = $\{(Q_n)\}$.

$(P_n), (Q_n)$: limit of centres.

We know $\bigcup_{n=1}^{\infty} Q_n \in \mathcal{F}$.

We want $\bigcup_{n=1}^{\infty} P_n \in \mathcal{F}$,

so shall prove $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$.

Problem

$$\begin{array}{ccc} \mathcal{K}^{\mathbb{N}}\text{-game} & & \mathcal{K}_{\nearrow}^{\mathbb{N}}\text{-game} \\ \left(\begin{array}{l} K_1 : \\ K_2 : \overset{1}{\bullet} \\ \vdots \\ K_{10} : \overset{2}{\bullet} \\ \vdots \\ K_{73} : \overset{3}{\bullet} \end{array} \right) & \longleftrightarrow & \left(\begin{array}{l} K_1 : \\ K_2 : \bullet \bullet \bullet \\ \vdots \\ K_{10} : \overset{2}{\bullet} \\ \vdots \\ K_{73} : \overset{3}{\bullet} \end{array} \right) \end{array}$$

The limit must belong to P_1 .