Sum formulas for multiple zeta values and symmetric multiple zeta values

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Japan Europe Number Theory
Exchange Seminar

16 November 2021

Joint work with Minoru Hirose and Hideki Murahara.

Reference: *Generating functions for sums of polynomial multiple zeta values*, Tohoku Math. J. (2), to appear, arXiv:2011.04220.

Sum formula for multiple zeta values

multiple zeta(-star) values

$$\zeta(k_1,\dots,k_r) := \sum_{1\le m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \in \mathbb{R}$$

$$\zeta^{\star}(k_1,\dots,k_r) := \sum_{1\le m_1\le\cdots\le m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \in \mathbb{R}$$

$\left(\begin{array}{l} r\ge 0 \\ k_1,\dots,k_r\ge 1 \\ k_r\ge 2 \end{array}\right)$

$\boxed{\text{Thm}}$ (sum formula for multiple zeta(-star) values)

$r\ge 0$, $w\ge r+2$.

Then

$\ge 2.$

$$\sum_{wt=w} \zeta(\underbrace{\mathbf{k}}_{r}, a) = \zeta(w), \quad \sum_{wt=w} \zeta^{\star}(\underbrace{\mathbf{k}}_{r}, a) = \binom{w-1}{r}\zeta(w)$$

total weight

(i.e. $k_1+\cdots+k_r+a$)

depth

(i.e. $\mathbf{k}=(k_1,\dots,k_r)$)

Symmetric multiple zeta values

harmonic regularization $\in \mathcal{Z}[T]$

↑ span{MZVs}

$$\zeta_S(k_1,\dots,k_r) := \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r}\, \zeta(k_1,\dots,k_i)\, \zeta(k_r,\dots,k_{i+1})$$

$(\zeta_S^{\circledast})$ $\qquad (\zeta^{\circledast}) \qquad (\zeta^{\circledast}) \qquad (k_1,\dots,k_r \geq 1)$

$\zeta_S^{(\circledast)} \bmod \zeta(2)\mathcal{Z}$: symmetric MZ(S)Vs (Kaneko–Zagier) $\qquad \zeta_S^{(\circledast)} \in \mathcal{Z}$

↑ span{MZVs}

$\boxed{\text{Thm}}$ (sum formula for symmetric MZVs, Murahara)

$r, s \geq 0, \quad w \geq r+s+2.$

Then

$$\sum_{wt=w} \zeta_S(\underbrace{\mathbf{k}}_{r}, a, \underbrace{\mathbf{l}}_{s}) \equiv \left(-(-1)^r\binom{w-1}{r} + (-1)^s\binom{w-1}{s}\right)\zeta(w).$$

$$\zeta_S^{\circledast} \qquad \equiv \left((-1)^s\binom{w-1}{r} - (-1)^r\binom{w-1}{s}\right)\zeta(w).$$

In particular, if $s=0$, then

$$\sum_{wt=w} \zeta_S(\underbrace{\mathbf{k}}_{r}, a) \equiv \left(-(-1)^r\binom{w-1}{r} + 1\right)\zeta(w)$$

$$\zeta_S^{\circledast} \qquad \equiv \left(\binom{w-1}{r} - (-1)^r\right)\zeta(w).$$

strikingly similar to the sum formula for MZVs

What I want to do in this talk

① to explain the similarity (to some extent)

(i.e. to deduce the sum formula for symmetric M2Vs
from " M2Vs
in a simple manner.)

② to give a common generalization of the two sum formulas
(in terms of polynomial M2Vs)

① [$s=0$ case]

$$\zeta_s(\underbrace{\mathbf{k}}_{r},a) = \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r+a}\,\zeta(k_1,\dots,k_i)\,\underbrace{\zeta(a,k_r,\dots,k_{i+1})}_{} + \zeta(\mathbf{k},a)$$

$\| \leftarrow$ antipode

$$\sum_{i'=i}^{r}(-1)^{r-i'}\zeta^\star(k_{i'+1},\dots,k_r,a)\,\zeta(k_{i'},\dots,k_{i+1})$$

$$= \sum_{i'=0}^{r}(-1)^{r-i'}(-1)^{k_{i'+1}+\cdots+k_r+a}\,\zeta^\star(k_{i'+1},\dots,k_r,a) \times \Big(\underbrace{\sum_{i=0}^{i'}(-1)^{k_{i+1}+\cdots+k_{i'}}\zeta(k_1,\dots,k_i)\,\zeta(k_{i'},\dots,k_{i+1})}_{=\ \zeta_s(k_1,\dots,k_{i'})}\Big) + \zeta(\mathbf{k},a)$$

$$= \sum_{i=0}^{r}(-1)^{r-i}(-1)^{k_{i+1}+\cdots+k_r+a}\,\zeta^\star(k_{i+1},\dots,k_r,a)\,\zeta_s(k_1,\dots,k_i) + \zeta(\mathbf{k},a)$$

Add this for all $(\mathbf{k},a)$ of total weight w:

$$\sum_{\mathrm{wt}=w}\zeta_s(\underbrace{\mathbf{k}}_{r},a) = \sum_{i=0}^{r}(-1)^{r-i}\sum_{w_1+w_2=w}\Big((-1)^{w_1}\sum_{\mathrm{wt}=w_1}\zeta^\star(\underbrace{\mathbf{k}}_{r-i},a)\Big)\underbrace{\Big(\sum_{\mathrm{wt}=w_2}\zeta_s(\underbrace{\mathbf{k}}_{i})\Big)}_{\equiv\ 0} + \sum_{\mathrm{wt}=w}\zeta(\underbrace{\mathbf{k}}_{r},a)$$

unless $i=0$ and $w_2=0$

$$\equiv (-1)^r(-1)^w\sum_{\mathrm{wt}=w}\zeta^\star(\underbrace{\mathbf{k}}_{r},a) + \sum_{\mathrm{wt}=w}\zeta(\underbrace{\mathbf{k}}_{r},a)$$

sum formula for M2Vs

$$\equiv \Big((-1)^r(-1)^w\binom{w-1}{r}+1\Big)\zeta(w) \equiv \Big(-(-1)^r\binom{w-1}{r}+1\Big)\zeta(w).$$

[general S] Need to use Schur MZVs of anti-hook type

$\boxed{\text{Def}}$

$$\zeta\left(\begin{array}{|ccc|c|}\cline{4-4} \multicolumn{3}{c|}{} & k_1 \\ \multicolumn{3}{c|}{} & \vdots \\ \multicolumn{3}{c|}{} & k_r \\ \hline l_1 & \cdots & l_s & a \\ \hline\end{array}\right) := \sum_{\substack{m_1 > \cdots > m_r > p \\ n_1 \le \cdots \le n_s \le p}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_1^{l_1} \cdots n_s^{l_s} p^a} \qquad (a \ge 2)$$

Then $\zeta\left(\begin{array}{|c|}\hline k_1 \\ \vdots \\ k_r \\ a \\ \hline\end{array}\right) = \zeta(k_1, \ldots, k_r, a)$, $\quad \zeta\left(\begin{array}{|ccc|}\hline l_1 \cdots & l_s & a \\ \hline\end{array}\right) = \zeta^{\star}(l_1, \ldots, l_s, a)$

$\boxed{\text{Prop}}$

$$\zeta_S(\underbrace{\mathbf{k}}_{r}, a, \underbrace{\mathbf{l}}_{s}) = \sum_{i=0}^{r} (-1)^{r-i} (-1)^{k_{i+1}+\cdots+k_r+a+l_1+\cdots+l_s} \zeta\left(\begin{array}{|ccc|c|}\cline{4-4} \multicolumn{3}{c|}{} & l_s \\ \multicolumn{3}{c|}{} & \vdots \\ \multicolumn{3}{c|}{} & l_1 \\ \hline k_{i+1} & \cdots & k_r & a \\ \hline\end{array}\right) \zeta_S(k_1, \ldots, k_i)$$

$$+ \sum_{j=0}^{s} (-1)^j \zeta\left(\begin{array}{|ccc|c|}\cline{4-4} \multicolumn{3}{c|}{} & k_1 \\ \multicolumn{3}{c|}{} & \vdots \\ \multicolumn{3}{c|}{} & k_r \\ \hline l_j & \cdots & l_1 & a \\ \hline\end{array}\right) \zeta_S(l_{j+1}, \ldots, l_s)$$

$\boxed{\text{Thm}}$ (Bachmann, Kadota, Suzuki, Yamamoto, Yamasaki)

$r, s \ge 0$, $w \ge r+s+2$.

Then
$$\sum_{\mathrm{wt}=w} \zeta\left(\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \mathbf{k} \\ \hline \underbrace{\mathbf{l}}_{s} & a \\ \hline\end{array}\right\}r\Bigg) = \binom{w-1}{s} \zeta(w)$$

← doesn't depend on r.

②

$\boxed{\text{Def}}$ (Polynomial MZVs ; Hirose, Murahara, Saito)

$$\zeta_{x,y}(\mathbf{k}) := \sum_{i=0}^{r} \zeta(k_1,\ldots,k_i)\,\zeta(k_r,\ldots,k_{i+1})\, x^{k_1+\cdots+k_i}\, y^{k_{i+1}+\cdots+k_r} \in \mathbb{Z}[T][x,y]$$

Then $\zeta_{1,-1} = \zeta_S$, $\zeta_{1,0} = \zeta$.

$\boxed{\text{Main Thm}}$ (sum formula for polynomial MZVs in terms of generating functions)

$$\sum_{\substack{\mathbf{k},\boldsymbol{l}\\ a\geq 2}} \zeta_{x,y}(\mathbf{k},a,\boldsymbol{l})\, A^{\mathrm{dep}\,\mathbf{k}} B^{\mathrm{dep}\,\boldsymbol{l}}\, W^{\mathrm{wt}\,\mathbf{k}+\mathrm{wt}\,\boldsymbol{l}+a}$$

$$= \frac{yW}{1-B}\Big(\psi_1(y(1-A)W) - \psi_1(y(B-A)W)\Big)\frac{\Gamma_1(xW)\,\Gamma_1(yW)}{\Gamma_1(x(1-A)W)\,\Gamma_1(y(1-A)W)}$$

$+$ (something similar)

where $\psi_1(W) := \sum_{k=2}^{\infty} \zeta(k) W^{k-1} \in \mathbb{Z}[W]$, $\Gamma_1(W) := \exp\Big(\sum_{k=1}^{\infty} \frac{\zeta(k)}{k} W^k\Big) \in \mathbb{Z}[T][[W]]$

$\boxed{\text{Cor}}$

$$\sum_{\mathrm{wt}=w} \zeta_{x,y}(\underbrace{\mathbf{k}}_{r}, a, \underbrace{\boldsymbol{l}}_{s}) \in \mathbb{Q}[T, \zeta(2),\ldots,\zeta(w)][x,y].$$

$\boxed{\text{Cor}}$

$x=1,\ y=0$:

$$\sum_{\substack{\mathbf{k},\mathbf{l}\\ a\geq 2}} \zeta(\mathbf{k},a,\mathbf{l})\, A^{\mathrm{dep}\,\mathbf{k}} B^{\mathrm{dep}\,\mathbf{l}}\, W^{\mathrm{wt}\,\mathbf{k}+\mathrm{wt}\,\mathbf{l}+a} = \frac{W}{1-A}\Big(\psi_1\big((1-B)W\big) - \psi_1\big((A-B)W\big)\Big)\frac{\Gamma_1(W)}{\Gamma_1\big((1-B)W\big)}$$

($B=0 \Rightarrow$ sum formula for MZVs)

$x=1,\ y=-1$:

$$\sum_{\substack{\mathbf{k},\mathbf{l}\\ a\geq 2}} \zeta_S(\mathbf{k},a,\mathbf{l})\, A^{\mathrm{dep}\,\mathbf{k}} B^{\mathrm{dep}\,\mathbf{l}}\, W^{\mathrm{wt}\,\mathbf{k}+\mathrm{wt}\,\mathbf{l}+a}$$

$$= -\frac{W}{1-B}\Big(\psi_1\big(-(1-A)W\big) - \psi_1\big((A-B)W\big)\Big)\underbrace{\frac{\pi W}{\sin \pi W}}_{\equiv\, 1}\cdot\underbrace{\frac{\sin \pi(1-A)W}{\pi(1-A)W}}_{\equiv\, 1}$$

$+$ (something similar)

$\equiv 1 \pmod{\zeta(2)}$

$$\Big(\Gamma_1(W)\Gamma_1(-W) = \underbrace{\frac{\pi W}{\sin \pi W}}_{\in\, \mathbb{Q}[[\pi^2 W^2]]}\Big)$$