

Typical \mathcal{F}_σ Sets and Typical Continuous Functions

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Typical Closed Sets

Work in $I = [0, 1]$.

$\mathcal{K} := \{K \subset I \mid K: \text{closed}\}$

equipped with the Hausdorff metric d

($d(K, \emptyset) := 1$ for nonempty $K \in \mathcal{K}$).

d defines residuality (comeagreness) on \mathcal{K} .

e.g.

$\{K \in \mathcal{K} \mid K: \text{null}\}$: residual in \mathcal{K} .

We say **typical** closed sets are null.

↑
all in some residual set

Typical \mathcal{F}_σ Sets

We need to define residuality on \mathcal{F}_σ .

Look at the surjection

$$\begin{array}{ccc} \mathcal{K}^{\mathbb{N}} & \longrightarrow & \mathcal{F}_\sigma \\ \Psi & & \Psi \\ (K_n) & \longmapsto & \bigcup_{n=1}^{\infty} K_n \end{array}$$

This map gives \mathcal{F}_σ a topology, hence residuality.

It turns out that for $\mathcal{F} \subset \mathcal{F}_\sigma$,

$$\mathcal{F}: \text{residual} \iff \mathcal{F} \supset \{F \in \mathcal{F}_\sigma \mid F: \text{dense}\}.$$

This residuality is not useful.

Instead of giving \mathcal{F}_σ a topology, we define residuality directly:

Def

$\mathcal{F} \subset \mathcal{F}_\sigma$ is **residual** if

$$\{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$$

is residual in $\mathcal{K}^{\mathbb{N}}$.

Prop

For a σ -ideal \mathcal{I} on I , T.F.A.E.:

- ① $\mathcal{I} \cap \mathcal{F}_\sigma$ is residual in \mathcal{F}_σ
(i.e. $F \in \mathcal{I}$ for typical $F \in \mathcal{F}_\sigma$);
- ② $\mathcal{I} \cap \mathcal{K}$ is residual in \mathcal{K}
(i.e. $K \in \mathcal{I}$ for typical closed K).

Typical continuous functions

$C(I) := \{f: I \longrightarrow \mathbb{R} \mid f: \text{continuous}\}$

equipped with the supremum norm.

$C(I)$ has residuality.

Our main theorem links

- ⊗ typical continuous functions and
- ⊗ typical \mathcal{F}_σ sets.

Def

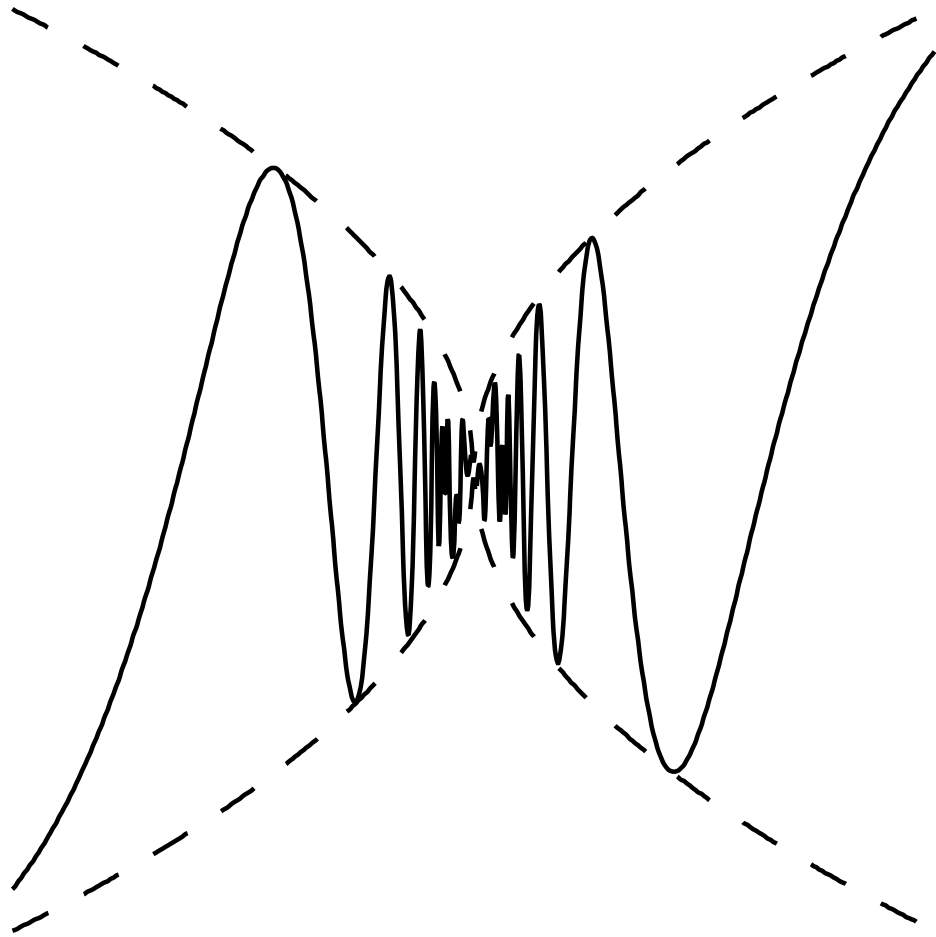
$x \in I$ is a **knot point** of $f \in C(I)$ if

$$D^+ f(x) := \limsup_{y \searrow x} \frac{f(y) - f(x)}{y - x} = \infty,$$

$$D_+ f(x) := \liminf_{y \searrow x} \quad \quad \quad = -\infty,$$

$$D^- f(x) := \limsup_{y \nearrow x} \quad \quad \quad = \infty,$$

$$D_- f(x) := \liminf_{y \nearrow x} \quad \quad \quad = -\infty.$$



For $f \in C(I)$,

$\{\text{knot points of } f\} \in \mathcal{G}_\delta$,

$N(f) := \{\text{non-knot points of } f\} \in \mathcal{F}_\sigma$.

Fact (unpublished)

For a σ -ideal \mathcal{I} on I , T.F.A.E.:

① $N(f) \in \mathcal{I}$ for typical $f \in C(I)$;

② $\mathcal{I} \cap \mathcal{K}$ is residual in \mathcal{K}

(i.e. $K \in \mathcal{I}$ for typical closed K).

① $\iff N(f) \in \mathcal{I} \cap \mathcal{F}_\sigma$ for typical $f \in C(I)$;

② $\iff \mathcal{I} \cap \mathcal{F}_\sigma$ is residual in \mathcal{F}_σ .

Main Thm

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

- ① $N(f) \in \mathcal{F}$ for typical $f \in C(I)$;
- ② \mathcal{F} is residual in \mathcal{F}_σ
(i.e. $F \in \mathcal{F}$ for typical $F \in \mathcal{F}_\sigma$).

Proof of Main Thm

Lem

We may find a 'good' $\mathbb{A} \subset \mathcal{K}^{\mathbb{N}} \times C(I)$ s.t.

① $((K_n), f) \in \mathbb{A}$ implies $\bigcup_{n=1}^{\infty} K_n = N(f)$;

② if $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}$ is residual, then

for typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A} \quad ((K_n), f) \in \mathbb{A}.$$

The proof of this lemma

① is very complicated and

② uses the Banach-Mazur game.

② \Rightarrow ①

Suppose $\mathcal{F} \subset \mathcal{F}_\sigma$ is residual.

$\mathcal{A} := \{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$: residual.

By Lem, for typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A} \quad ((K_n), f) \in \mathbb{A},$$

$$\therefore N(f) = \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}.$$

① \Rightarrow ②

Suppose $N(f) \in \mathcal{F}$ for typical $f \in C(I)$.

Take a dense \mathcal{G}_δ set $G \subset \{f \in C(I) \mid N(f) \in \mathcal{F}\}$.

$\mathcal{A} := \{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \exists f \in G \quad ((K_n), f) \in \mathbb{A}\}$.

$(K_n) \in \mathcal{A}$ implies $\bigcup_{n=1}^{\infty} K_n = N(f) \in \mathcal{F}$.

Thus it suffices to show that \mathcal{A} is residual.

\mathcal{A} turns out to be analytic (since \mathbb{A} is ‘good’).

$\therefore \mathcal{A}$ has the Baire Property.

$\therefore \mathcal{A}$ is either meagre or residual

(topological 0-1 law).

Suppose \mathcal{A} is meagre.

Then \mathcal{A}^c is residual.

By Lem, for typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A}^c \quad ((K_n), f) \in \mathbb{A}.$$

Thus

$$\exists f \in G \exists (K_n) \in \mathcal{A}^c \quad ((K_n), f) \in \mathbb{A}.$$

This contradicts the definition of \mathcal{A} . ■