Typical $\mathcal{F}_\sigma$ Sets and Typical Continuous Functions

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Typical Closed Sets

Work in $I = [0,1]$.

$\mathcal{K} := \{K \subset I \mid K\text{: closed}\}$

equipped with the Hausdorff metric d

$(d(K, \emptyset) := 1$ for nonempty $K \in \mathcal{K})$.

d defines residuality (comeagreness) on $\mathcal{K}$.

e.g.

$\{K \in \mathcal{K} \mid K\text{: null}\}$: residual in $\mathcal{K}$.

We say **typical** closed sets are null.

↑

all in some residual set

Typical $\mathcal{F}_\sigma$ Sets

We need to define residuality on $\mathcal{F}_\sigma$.

Look at the surjection

$$\begin{array}{ccc} \mathcal{K}^{\mathbb{N}} & \longrightarrow & \mathcal{F}_\sigma \\ \Cup & & \Cup \\ (K_n) & \longmapsto & \bigcup_{n=1}^{\infty} K_n \end{array}$$

This map gives $\mathcal{F}_\sigma$ a topology, hence residuality.

It turns out that for $\mathcal{F} \subset \mathcal{F}_\sigma$,

$$\mathcal{F}\text{: residual} \iff \mathcal{F} \supset \{F \in \mathcal{F}_\sigma \mid F\text{: dense}\}.$$

This residuality is not useful.

Instead of giving $\mathcal{F}_\sigma$ a topology, we define residuality directly:

Def

$\mathcal{F} \subset \mathcal{F}_\sigma$ is **residual** if

$$\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \,\middle|\, \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}$$

is residual in $\mathcal{K}^{\mathbb{N}}$.

Prop

For a σ-ideal $\mathcal{I}$ on I, T.F.A.E.:

① $\mathcal{I} \cap \mathcal{F}_\sigma$ is residual in $\mathcal{F}_\sigma$

(i.e. $F \in \mathcal{I}$ for typical $F \in \mathcal{F}_\sigma$);

② $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$

(i.e. $K \in \mathcal{I}$ for typical closed K).

Typical continuous functions

$C(I) := \{f : I \longrightarrow \mathbb{R} \mid f\text{: continuous}\}$

equipped with the supremum norm.

$C(I)$ has residuality.

Our main theorem links

- typical continuous functions and
- typical $\mathcal{F}_\sigma$ sets.

Def

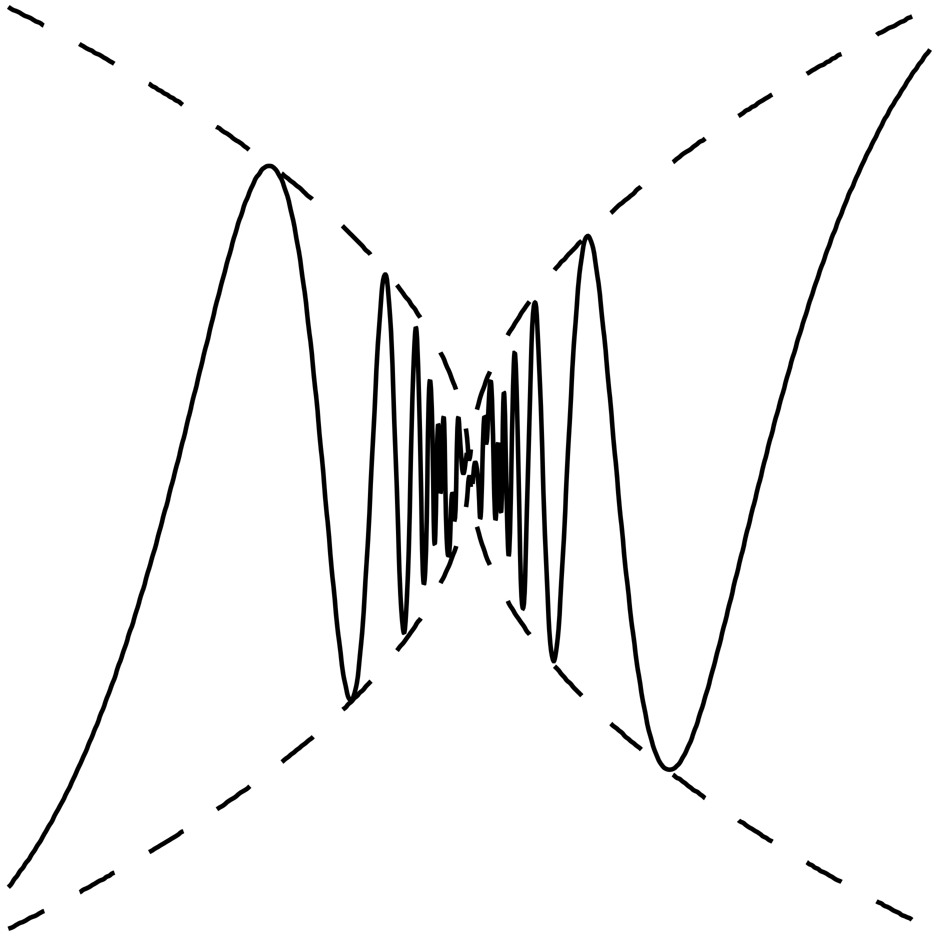
$x \in I$ is a **knot point** of $f \in C(I)$ if

$$D^+ f(x) := \limsup_{y \searrow x} \frac{f(y) - f(x)}{y - x} = \infty,$$

$$D_+ f(x) := \liminf_{y \searrow x} \qquad\qquad = -\infty,$$

$$D^- f(x) := \limsup_{y \nearrow x} \qquad\qquad = \infty,$$

$$D_- f(x) := \liminf_{y \nearrow x} \qquad\qquad = -\infty.$$



For $f \in C(I)$,

$$\{\text{knot points of } f\} \in \mathcal{G}_\delta,$$

$$N(f) := \{\text{non-knot points of } f\} \in \mathcal{F}_\sigma.$$

Fact (unpublished)

For a σ-ideal $\mathcal{I}$ on I, T.F.A.E.:

① $N(f) \in \mathcal{I}$ for typical $f \in C(I)$;

② $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$

(i.e. $K \in \mathcal{I}$ for typical closed K).

① $\iff N(f) \in \mathcal{I} \cap \mathcal{F}_\sigma$ for typical $f \in C(I)$;

② $\iff \mathcal{I} \cap \mathcal{F}_\sigma$ is residual in $\mathcal{F}_\sigma$.

Main Thm

For $\mathcal{F} \subset \mathcal{F}_\sigma$, T.F.A.E.:

① $N(f) \in \mathcal{F}$ for typical $f \in C(I)$;

② $\mathcal{F}$ is residual in $\mathcal{F}_\sigma$

(i.e. $F \in \mathcal{F}$ for typical $F \in \mathcal{F}_\sigma$).

Proof of Main Thm

Lem

We may find a 'good' $\mathbb{A} \subset \mathcal{K}^{\mathbb{N}} \times C(I)$ s.t.

- $\big((K_n), f\big) \in \mathbb{A}$ implies $\bigcup_{n=1}^{\infty} K_n = N(f)$;
- if $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}$ is residual, then for typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A} \quad \big((K_n), f\big) \in \mathbb{A}.$$

The proof of this lemma

- is very complicated and
- uses the Banach-Mazur game.

② ⇒ ①

Suppose $\mathcal{F} \subset \mathcal{F}_\sigma$ is residual.

$\mathcal{A} := \left\{(K_n) \in \mathcal{K}^{\mathbb{N}} \,\middle|\, \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\right\}$: residual.

By Lem, for typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A} \quad \big((K_n), f\big) \in \mathbb{A},$$

$$\therefore N(f) = \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}.$$

① ⇒ ②

Suppose $N(f) \in \mathcal{F}$ for typical $f \in C(I)$.
Take a dense $\mathcal{G}_\delta$ set $G \subset \{f \in C(I) \mid N(f) \in \mathcal{F}\}$.
$\mathcal{A} := \{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \exists f \in G \quad ((K_n), f) \in \mathbb{A}\}$.

$(K_n) \in \mathcal{A}$ implies $\bigcup_{n=1}^{\infty} K_n = N(f) \in \mathcal{F}$.
Thus it suffices to show that $\mathcal{A}$ is residual.

$\mathcal{A}$ turns out to be analytic (since $\mathbb{A}$ is 'good').
$\therefore$ $\mathcal{A}$ has the Baire Property.
$\therefore$ $\mathcal{A}$ is either meagre or residual
(topological 0-1 law).

Suppose $\mathcal{A}$ is meagre.
Then $\mathcal{A}^c$ is residual.

By Lem, for typical $f \in C(I)$,

$$\exists (K_n) \in \mathcal{A}^c \quad \big((K_n), f\big) \in \mathbb{A}.$$

Thus

$$\exists f \in G \ \exists (K_n) \in \mathcal{A}^c \quad \big((K_n), f\big) \in \mathbb{A}.$$

This contradicts the definition of $\mathcal{A}$. $\blacksquare$