

1 **RESTRICTED SUM FORMULA FOR FINITE AND SYMMETRIC**  
 2 **MULTIPLE ZETA VALUES**

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ABSTRACT. The sum formula for finite and symmetric multiple zeta values, established by Wakabayashi and the authors, implies that if the weight and depth are fixed and the specified component is required to be more than one, then the values sum up to a rational multiple of the analogue of the Riemann zeta value. We prove that the result remains true if we further demand that the component should be more than two or that another component should also be more than one.

4 1. INTRODUCTION

The *multiple zeta values* and *multiple zeta-star values* are the real numbers defined by

$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r \geq 1} \frac{1}{m_1^{k_1} \dots m_r^{k_r}},$$

$$\zeta^*(k_1, \dots, k_r) = \sum_{m_1 \geq \dots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}$$

5 for  $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$  with  $k_1 \geq 2$ . They are generalisations of the values of the  
 6 Riemann zeta function at positive integers, and they are known to have interesting  
 7 algebraic structures due to the many relations among them, the simplest being  
 8  $\zeta(2, 1) = \zeta(3)$ . See, for example, the book [9] by Zhao for further details on multiple  
 9 zeta(-star) values.

The variants of multiple zeta values that we shall be looking at in this paper are *finite multiple zeta values*  $\zeta_{\mathcal{A}}(k_1, \dots, k_r)$  and *symmetric multiple zeta values*  $\zeta_{\mathcal{S}}(k_1, \dots, k_r)$  (the latter also known as symmetrised multiple zeta values and finite real multiple zeta values), both introduced by Kaneko and Zagier [4] (see [9] for details). Set  $\mathcal{A} = \prod_p \mathbb{F}_p / \bigoplus_p \mathbb{F}_p$ , where  $p$  runs over all primes. For  $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ , we define

$$\zeta_{\mathcal{A}}(k_1, \dots, k_r) = \left( \sum_{p > m_1 > \dots > m_r \geq 1} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \bmod p \right)_p \in \mathcal{A},$$

$$\zeta_{\mathcal{A}}^*(k_1, \dots, k_r) = \left( \sum_{p > m_1 \geq \dots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \bmod p \right)_p \in \mathcal{A}.$$

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Let  $\mathcal{Z}$  denote the  $\mathbb{Q}$ -linear subspace of  $\mathbb{R}$  spanned by the multiple zeta values. For  $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ , we define

$$\zeta_{\mathcal{S}}(k_1, \dots, k_r) = \sum_{j=0}^r (-1)^{k_1 + \dots + k_j} \zeta(k_j, \dots, k_1) \zeta(k_{j+1}, \dots, k_r) \pmod{\zeta(2)} \in \mathcal{Z}/\zeta(2)\mathcal{Z},$$

$$\zeta_{\mathcal{S}}^*(k_1, \dots, k_r) = \sum_{j=0}^r (-1)^{k_1 + \dots + k_j} \zeta^*(k_j, \dots, k_1) \zeta^*(k_{j+1}, \dots, k_r) \pmod{\zeta(2)} \in \mathcal{Z}/\zeta(2)\mathcal{Z},$$

1 where we set  $\zeta(\emptyset) = \zeta^*(\emptyset) = 1$ . The multiple zeta(-star) values that appear in the  
 2 definition of the symmetric multiple zeta(-star) values are the regularised values  
 3 if the first component is 1; although there are two ways of regularisation, called  
 4 the harmonic regularisation and the shuffle regularisation, it is known that the  
 5 symmetric multiple zeta values remain unchanged as elements of  $\mathcal{Z}/\zeta(2)\mathcal{Z}$  no matter  
 6 which regularisation we use (see [4]).

7 Kaneko and Zagier [4] made a striking conjecture that the finite multiple zeta  
 8 values and the symmetric multiple zeta values are isomorphic; more precisely, if  
 9 we let  $\mathcal{Z}_{\mathcal{A}}$  denote the  $\mathbb{Q}$ -linear subspace of  $\mathcal{A}$  spanned by the finite multiple zeta  
 10 values, then  $\mathcal{Z}_{\mathcal{A}}$  and  $\mathcal{Z}/\zeta(2)\mathcal{Z}$  are isomorphic as  $\mathbb{Q}$ -algebras via the correspondence  
 11  $\zeta_{\mathcal{A}}(k_1, \dots, k_r) \leftrightarrow \zeta_{\mathcal{S}}(k_1, \dots, k_r)$ . It means that  $\zeta_{\mathcal{A}}(k_1, \dots, k_r)$  and  $\zeta_{\mathcal{S}}(k_1, \dots, k_r)$   
 12 satisfy the same relations, and a notable example of such relations is the sum  
 13 formula (Theorem 1.1). In what follows, we use the letter  $\mathcal{F}$  when it can be replaced  
 14 with either  $\mathcal{A}$  or  $\mathcal{S}$ ; for example, by  $\zeta_{\mathcal{F}}(1) = 0$  we mean that both  $\zeta_{\mathcal{A}}(1) = 0$  and  
 15  $\zeta_{\mathcal{S}}(1) = 0$  are true. We write

$$\mathfrak{z}_{\mathcal{F}}(k) = \begin{cases} (B_{p-k}/k \pmod{p})_p & \text{if } \mathcal{F} = \mathcal{A}; \\ \zeta(k) \pmod{\zeta(2)} & \text{if } \mathcal{F} = \mathcal{S} \end{cases}$$

16 for  $k \in \mathbb{Z}_{\geq 2}$ , where  $B_n$  denotes the  $n$ -th Bernoulli number. Note that it can be ver-  
 17 ified rather easily that  $\zeta_{\mathcal{F}}(k-1, 1) = \mathfrak{z}_{\mathcal{F}}(k)$  for  $k \in \mathbb{Z}_{\geq 2}$ , so that  $(B_{p-k}/k \pmod{p})_p$   
 18 corresponds to  $\zeta(k) \pmod{\zeta(2)}$  via the above-mentioned isomorphism  $\mathcal{Z}_{\mathcal{A}} \cong$   
 19  $\mathcal{Z}/\zeta(2)\mathcal{Z}$ .

**Theorem 1.1** (Saito-Wakabayashi [8], Murahara [5]). *For  $k, r, i \in \mathbb{Z}$  with  $1 \leq i \leq r \leq k-1$ , we have*

$$\begin{aligned} \sum_{\substack{k_1 + \dots + k_r = k \\ k_i \geq 2}} \zeta_{\mathcal{F}}(k_1, \dots, k_r) &= (-1)^r \sum_{\substack{k_1 + \dots + k_r = k \\ k_i \geq 2}} \zeta_{\mathcal{F}}^*(k_1, \dots, k_r) \\ &= (-1)^{i-1} \left( \binom{k-1}{i-1} + (-1)^r \binom{k-1}{r-i} \right) \mathfrak{z}_{\mathcal{F}}(k). \end{aligned}$$

20 The theorem implies that the sums belong to  $\mathbb{Q}\mathfrak{z}_{\mathcal{F}}(k)$ . Our main theorem states  
 21 that similar sums also belong to  $\mathbb{Q}\mathfrak{z}_{\mathcal{F}}(k)$  if  $k$  is odd:

22 **Theorem 1.2** (Main theorem). *Let  $k$  be an odd integer with  $k \geq 3$ , and let  $r$  be*  
 23 *an integer with  $1 \leq r \leq k-2$ .*

24 (1) *For  $i \in \mathbb{Z}$  with  $1 \leq i \leq r$ , we have*

$$\sum_{\substack{k_1 + \dots + k_r = k \\ k_i \geq 3}} \zeta_{\mathcal{F}}(k_1, \dots, k_r) = (-1)^r \sum_{\substack{k_1 + \dots + k_r = k \\ k_i \geq 3}} \zeta_{\mathcal{F}}^*(k_1, \dots, k_r) \in \mathbb{Q}\mathfrak{z}_{\mathcal{F}}(k).$$

1 (2) For distinct  $i, j \in \mathbb{Z}$  with  $1 \leq i, j \leq r$ , we have

$$\sum_{\substack{k_1 + \dots + k_r = k \\ k_i, k_j \geq 2}} \zeta_{\mathcal{F}}(k_1, \dots, k_r) = (-1)^r \sum_{\substack{k_1 + \dots + k_r = k \\ k_i, k_j \geq 2}} \zeta_{\mathcal{F}}^*(k_1, \dots, k_r) \in \mathbb{Q}\mathfrak{Z}_{\mathcal{F}}(k).$$

2 The rational coefficients can be written explicitly, though in a rather compli-  
 3 cated manner, in terms of binomial coefficients (see Theorem 3.1 for the precise  
 4 statement).

5 *Remark 1.3.* If  $k$  is even, then  $\mathfrak{Z}_{\mathcal{F}}(k) = 0$  and numerical experiments suggest that  
 6 the sums are not always equal to 0.

## 7 2. PRELIMINARY LEMMAS

8 This section will give a few preliminary lemmas that will be used to prove our  
 9 main theorem in the next section.

10 An *index* is a (possibly empty) sequence of positive integers. For an index  
 11  $\mathbf{k} = (k_1, \dots, k_r)$ , the number  $r$  is called its *depth* and  $k_1 + \dots + k_r$  its *weight*.

12 **Proposition 2.1.** *If  $(k_1, \dots, k_r)$  is a nonempty index, then*

$$\sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{F}}(k_{\sigma(1)}, \dots, k_{\sigma(r)}) = \sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{F}}^*(k_{\sigma(1)}, \dots, k_{\sigma(r)}) = 0,$$

13 where  $\mathfrak{S}_r$  denotes the symmetric group of order  $r$ .

14 *Proof.* Roughly speaking, the sums are zero because they can be written as poly-  
 15 nomials of the values  $\zeta_{\mathcal{F}}(k)$ , which are all zero. For details, see [1, Theorem 2.3]  
 16 and [7, Proposition 2.7], for example.  $\square$

17 We write  $\{k\}^r$  for the  $r$  times repetition of  $k$ .

18 **Corollary 2.2.** *For  $k, r \in \mathbb{Z}_{\geq 1}$ , we have*

$$\zeta_{\mathcal{F}}(\{k\}^r) = \zeta_{\mathcal{F}}^*(\{k\}^r) = 0.$$

19 *Proof.* Apply Proposition 2.1 to  $(k_1, \dots, k_r) = (\{k\}^r)$ .  $\square$

20 **Definition 2.3.** For each index  $\mathbf{k}$ , write its components as sums of ones, and define  
 21 its *Hoffman dual*  $\mathbf{k}^{\vee}$  as the index obtained by swapping plus signs and commas.

22 **Example 2.4.** If  $\mathbf{k} = (2, 1, 3) = (1 + 1, 1, 1 + 1 + 1)$ , then  $\mathbf{k}^{\vee} = (1, 1 + 1 + 1, 1, 1) =$   
 23  $(1, 3, 1, 1)$ .

24 The following theorem, known as *duality*, was proved by Hoffman [1] for the  
 25  $\mathcal{F} = \mathcal{A}$  case and by Jarossay [2] for the  $\mathcal{F} = \mathcal{S}$  case:

26 **Theorem 2.5** (Hoffman [1], Jarossay [2]). *If  $\mathbf{k}$  is a nonempty index, then*

$$\zeta_{\mathcal{F}}^*(\mathbf{k}^{\vee}) = -\zeta_{\mathcal{F}}^*(\mathbf{k}).$$

27 For indices  $\mathbf{k}$  and  $\mathbf{l}$  of the same weight, we write  $\mathbf{k} \preceq \mathbf{l}$  to mean that, writing their  
 28 components as sums of ones, we can obtain  $\mathbf{l}$  from  $\mathbf{k}$  by replacing some (possibly  
 29 none) of the plus signs with commas. For example,  $(2, 1, 3) = (1 + 1, 1, 1 + 1 + 1) \preceq$   
 30  $(1, 1, 1, 1 + 1, 1) = (1, 1, 1, 2, 1)$ .

31 **Corollary 2.6.** *If  $\mathbf{k}$  is a nonempty index of depth  $r$ , then*

$$(-1)^r \zeta_{\mathcal{F}}(\mathbf{k}) = \sum_{\mathbf{l} \succeq \mathbf{k}} \zeta_{\mathcal{F}}(\mathbf{l}).$$

1 *Proof.* An easy combinatorial argument shows that this corollary is equivalent to  
 2 Theorem 2.5; see [7, Corollary 2.15] for details.  $\square$

3 We adopt the standard convention for binomial coefficients that  $\binom{a}{b} = 0$  if  $a \in$   
 4  $\mathbb{Z}_{\geq 0}$  and  $b \in \mathbb{Z} \setminus \{0, \dots, a\}$ . For notational simplicity, we write

$$\begin{bmatrix} a \\ b \end{bmatrix} = (-1)^b \binom{a}{b}$$

5 for  $a \in \mathbb{Z}_{\geq 0}$  and  $b \in \mathbb{Z}$  (not to be confused with the Stirling numbers of the first  
 6 kind). Then Theorem 1.1 can be rewritten as follows:

**Theorem 2.7** (Another form of Theorem 1.1). *For  $k, r, i \in \mathbb{Z}$  with  $1 \leq i \leq r \leq$   
 $k - 1$ , we have*

$$\begin{aligned} \sum_{\substack{k_1 + \dots + k_r = k \\ k_i \geq 2}} \zeta_{\mathcal{F}}(k_1, \dots, k_r) &= (-1)^r \sum_{\substack{k_1 + \dots + k_r = k \\ k_i \geq 2}} \zeta_{\mathcal{F}}^*(k_1, \dots, k_r) \\ &= \left( \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} - \begin{bmatrix} k-1 \\ r-i \end{bmatrix} \right) \mathfrak{Z}_{\mathcal{F}}(k). \end{aligned}$$

7 **Lemma 2.8.** *For  $a, b \in \mathbb{Z}_{\geq 0}$  with  $a + b$  odd, we have*

$$\zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^b) = - \begin{bmatrix} a+b+2 \\ a+1 \end{bmatrix} \mathfrak{Z}_{\mathcal{F}}(a+b+2) = \begin{bmatrix} a+b+2 \\ b+1 \end{bmatrix} \mathfrak{Z}_{\mathcal{F}}(a+b+2).$$

8 *Proof.* Applying Theorem 2.7 to  $k = a + b + 2$ ,  $r = a + b + 1$ , and  $i = a + 1$  gives

$$\zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^b) = \left( \begin{bmatrix} a+b+1 \\ a \end{bmatrix} - \begin{bmatrix} a+b+1 \\ b \end{bmatrix} \right) \mathfrak{Z}_{\mathcal{F}}(a+b+2),$$

and we have

$$\begin{aligned} \begin{bmatrix} a+b+1 \\ a \end{bmatrix} - \begin{bmatrix} a+b+1 \\ b \end{bmatrix} &= (-1)^a \binom{a+b+1}{a} - (-1)^b \binom{a+b+1}{b} \\ &= -(-1)^{a+1} \left( \binom{a+b+1}{a} + \binom{a+b+1}{a+1} \right) \\ &= -(-1)^{a+1} \binom{a+b+2}{a+1} \\ &= - \begin{bmatrix} a+b+2 \\ a+1 \end{bmatrix}. \end{aligned}$$

9 By a similar reasoning, we also have

$$\begin{bmatrix} a+b+1 \\ a \end{bmatrix} - \begin{bmatrix} a+b+1 \\ b \end{bmatrix} = \begin{bmatrix} a+b+2 \\ b+1 \end{bmatrix}. \quad \square$$

10 **Lemma 2.9.** *For  $a, b \in \mathbb{Z}_{\geq 0}$  and  $c \in \mathbb{Z}_{\geq -1}$  with  $a + b + c$  odd, we have*

$$\zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^c, 2, \{1\}^b) = \frac{1}{2} \left( \begin{bmatrix} a+b+c+4 \\ a+1 \end{bmatrix} - \begin{bmatrix} a+b+c+4 \\ b+1 \end{bmatrix} \right) \mathfrak{Z}_{\mathcal{F}}(a+b+c+4),$$

11 *where we understand that  $\zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^{-1}, 2, \{1\}^b) = \zeta_{\mathcal{F}}(\{1\}^a, 3, \{1\}^b)$ .*

*Proof.* Keeping Corollary 2.2 in mind, we apply Corollary 2.6 to  $\mathbf{k} = (\{1\}^a, 2, \{1\}^c, 2, \{1\}^b)$  to get

$$\begin{aligned} & -\zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^c, 2, \{1\}^b) \\ & = \zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^c, 2, \{1\}^b) + \zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^{b+c+2}) + \zeta_{\mathcal{F}}(\{1\}^{a+c+2}, 2, \{1\}^b), \end{aligned}$$

no matter whether  $c = -1$  or  $c \geq 0$ . This, together with Lemma 2.8, gives

$$\begin{aligned} & \zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^c, 2, \{1\}^b) \\ & = -\frac{1}{2}(\zeta_{\mathcal{F}}(\{1\}^a, 2, \{1\}^{b+c+2}) + \zeta_{\mathcal{F}}(\{1\}^{a+c+2}, 2, \{1\}^b)) \\ & = \frac{1}{2} \left( \begin{bmatrix} a+b+c+4 \\ a+1 \end{bmatrix} - \begin{bmatrix} a+b+c+4 \\ b+1 \end{bmatrix} \right) \mathfrak{Z}_{\mathcal{F}}(a+b+c+4). \quad \square \end{aligned}$$

### 1 3. PROOF OF THE MAIN THEOREM

2 Throughout this section, let  $k$  be an odd integer with  $k \geq 3$ , and let  $r, i, j$  be  
3 integers with  $1 \leq i \leq j \leq r \leq k-2$ . Set

$$I_{k,r,i,j} = \begin{cases} \{(k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r \mid k_i \geq 3\} & \text{if } i = j; \\ \{(k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r \mid k_i, k_j \geq 2\} & \text{if } i < j, \end{cases}$$

4 and write

$$S_{k,r,i,j} = \sum_{\mathbf{k} \in I_{k,r,i,j}} \zeta_{\mathcal{F}}(\mathbf{k}), \quad S_{k,r,i,j}^* = \sum_{\mathbf{k} \in I_{k,r,i,j}} \zeta_{\mathcal{F}}^*(\mathbf{k}).$$

5 For notational simplicity, we put  $i' = j - i + 1$ ,  $i'' = r - j + 1$ , and  $k' = k - r - 2$ ,  
6 so that  $i + i' + i'' + k' = k$ .

7 The aim of this section is to prove the following theorem, from which Theorem 1.2  
8 easily follows:

9 **Theorem 3.1.** *We have*

$$S_{k,r,i,j} = (-1)^r S_{k,r,i,j}^* = \frac{1}{2} N_{k,r,i,j} \mathfrak{Z}_{\mathcal{F}}(k),$$

where  $N_{k,r,i,j}$  is an integer given by

$$\begin{aligned} N_{k,r,i,j} & = (k' + i + 1) \left( \begin{bmatrix} k-1 \\ k'+i \end{bmatrix} - \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} \right) - (k' + i'' + 1) \left( \begin{bmatrix} k-1 \\ k'+i'' \end{bmatrix} - \begin{bmatrix} k-1 \\ i''-1 \end{bmatrix} \right) \\ & + k \left( \begin{bmatrix} k-2 \\ k'+i-1 \end{bmatrix} - \begin{bmatrix} k-2 \\ i-2 \end{bmatrix} - \begin{bmatrix} k-2 \\ k'+i''-1 \end{bmatrix} + \begin{bmatrix} k-2 \\ i''-2 \end{bmatrix} \right). \end{aligned}$$

10 **3.1. Proof that**  $S_{k,r,i,j} = (-1)^r S_{k,r,i,j}^*$ . In this subsection, we shall prove that  
11  $S_{k,r,i,j} = (-1)^r S_{k,r,i,j}^*$  (Lemma 3.4).

**Proposition 3.2.** *If  $(k_1, \dots, k_r)$  is an index, then*

$$\begin{aligned} \zeta_{\mathcal{F}}(k_r, \dots, k_1) & = (-1)^{k_1 + \dots + k_r} \zeta_{\mathcal{F}}(k_1, \dots, k_r), \\ \zeta_{\mathcal{F}}^*(k_r, \dots, k_1) & = (-1)^{k_1 + \dots + k_r} \zeta_{\mathcal{F}}^*(k_1, \dots, k_r). \end{aligned}$$

12 *Proof.* Easy from the definitions; see [7, Proposition 2.6] for details.  $\square$

13 **Proposition 3.3.** *If  $\mathbf{k} = (k_1, \dots, k_r)$  is a nonempty index, then*

$$\sum_{s=0}^r (-1)^s \zeta_{\mathcal{F}}^*(k_s, \dots, k_1) \zeta_{\mathcal{F}}(k_{s+1}, \dots, k_r) = 0,$$

1 where we set  $\zeta_{\mathcal{F}}(\emptyset) = \zeta_{\mathcal{F}}^*(\emptyset) = 1$ .

2 *Proof.* Well known; see [7, Proposition 2.9] for the detailed proof.  $\square$

3 **Lemma 3.4.** *We have*

$$S_{k,r,i,j} = (-1)^r S_{k,r,i,j}^*$$

4 *Proof.* Adding the equation in Proposition 3.3 for all  $(k_1, \dots, k_r) \in I_{k,r,i,j}$  gives

$$\sum_{s=0}^r (-1)^s \sum_{(k_1, \dots, k_r) \in I_{k,r,i,j}} \zeta_{\mathcal{F}}^*(k_s, \dots, k_1) \zeta_{\mathcal{F}}(k_{s+1}, \dots, k_r) = 0,$$

whose left-hand side we shall write as  $\sum_{s=0}^r (-1)^s A_s$  for simplicity. Observe that  $A_0 = S_{k,r,i,j}$  and that

$$\begin{aligned} A_r &= \sum_{(k_1, \dots, k_r) \in I_{k,r,i,j}} \zeta_{\mathcal{F}}^*(k_r, \dots, k_1) \\ &= \sum_{(k_1, \dots, k_r) \in I_{k,r,i,j}} (-1)^{k_1 + \dots + k_r} \zeta_{\mathcal{F}}^*(k_1, \dots, k_r) \\ &= -S_{k,r,i,j}^* \end{aligned}$$

by Proposition 3.2 because  $k$  is odd. For  $s = j, \dots, r-1$ , we have

$$\begin{aligned} A_s &= \sum_{l=0}^k \left( \sum_{(k_1, \dots, k_s) \in I_{l,s,i,j}} \zeta_{\mathcal{F}}^*(k_s, \dots, k_1) \right) \left( \sum_{k_{s+1} + \dots + k_r = k-l} \zeta_{\mathcal{F}}(k_{s+1}, \dots, k_r) \right) \\ &= 0 \end{aligned}$$

because of Proposition 2.1; we similarly have  $A_s = 0$  for  $s = 1, \dots, i-1$ . If  $i < j$  and  $i \leq s \leq j-1$ , then we have

$$\begin{aligned} A_s &= \sum_{l=0}^k \left( \sum_{\substack{k_1 + \dots + k_s = l \\ k_i \geq 2}} \zeta_{\mathcal{F}}^*(k_s, \dots, k_1) \right) \left( \sum_{\substack{k_{s+1} + \dots + k_r = k-l \\ k_j \geq 2}} \zeta_{\mathcal{F}}(k_{s+1}, \dots, k_r) \right) \\ &= \sum_{l=0}^k \left( (-1)^l \sum_{\substack{k_1 + \dots + k_s = l \\ k_i \geq 2}} \zeta_{\mathcal{F}}^*(k_1, \dots, k_s) \right) \left( \sum_{\substack{k_{s+1} + \dots + k_r = k-l \\ k_j \geq 2}} \zeta_{\mathcal{F}}(k_{s+1}, \dots, k_r) \right) \\ &= \sum_{l=0}^k (-1)^{l+s} \left( \begin{bmatrix} l-1 \\ i-1 \end{bmatrix} - \begin{bmatrix} l-1 \\ s-i \end{bmatrix} \right) \mathfrak{Z}_{\mathcal{F}}(l) \left( \begin{bmatrix} k-l-1 \\ j-s-1 \end{bmatrix} - \begin{bmatrix} k-l-1 \\ r-j \end{bmatrix} \right) \mathfrak{Z}_{\mathcal{F}}(k-l) \end{aligned}$$

5 by Proposition 3.2 and Theorem 2.7; since  $k$  is odd, either  $l$  or  $k-l$  must even  
6 and so  $\mathfrak{Z}_{\mathcal{F}}(l)\mathfrak{Z}_{\mathcal{F}}(k-l) = 0$  for all  $l = 0, \dots, k$ , from which it follows that  $A_s = 0$ .

7 Therefore we have  $S_{k,r,i,j} - (-1)^r S_{k,r,i,j}^* = 0$ , and the lemma follows.  $\square$

8 **3.2. Computation of  $S_{k,r,i,j}$ .** In this subsection, we shall compute  $S_{k,r,i,j}$  (Lemma 3.9).

9 The main ingredient of the computation is the following Ohno type relation, con-  
10 jectured by Kaneko [3] and established by Oyama [6]:

11 **Theorem 3.5** (Oyama [6, Theorem 1.4]). *Let  $\mathbf{k} = (k_1, \dots, k_r)$  be an index, and*  
12 *write its Hoffman dual as  $\mathbf{k}^\vee = (k'_1, \dots, k'_{r'})$ . Then for  $m \in \mathbb{Z}_{\geq 0}$ , we have*

$$\sum_{\substack{e_1 + \dots + e_r = m \\ e_1, \dots, e_r \geq 0}} \zeta_{\mathcal{F}}(k_1 + e_1, \dots, k_r + e_r) = \sum_{\substack{e'_1 + \dots + e'_{r'} = m \\ e'_1, \dots, e'_{r'} \geq 0}} \zeta_{\mathcal{F}}((k'_1 + e'_1, \dots, k'_{r'} + e'_{r'})^\vee).$$

1 **Lemma 3.6.** *We have*

$$S_{k,r,i,j} = \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \zeta_{\mathcal{F}}((i+e'_1, i'+e'_2, i''+e'_3)^\vee).$$

*Proof.* Theorem 3.5 shows that if  $i = j$ , then

$$\begin{aligned} S_{k,r,i,j} &= \sum_{\substack{e_1+\dots+e_r=k' \\ e_1, \dots, e_r \geq 0}} \zeta_{\mathcal{F}}(1+e_1, \dots, 1+e_{i-1}, 3+e_i, 1+e_{i+1}, \dots, 1+e_r) \\ &= \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \zeta_{\mathcal{F}}((i+e'_1, i'+e'_2, i''+e'_3)^\vee), \end{aligned}$$

and that if  $i < j$ , then

$$\begin{aligned} S_{k,r,i,j} &= \sum_{\substack{e_1+\dots+e_r=k' \\ e_1, \dots, e_r \geq 0}} \zeta_{\mathcal{F}}(1+e_1, \dots, 1+e_{i-1}, 2+e_i, 1+e_{i+1}, \dots, 1+e_{j-1}, 2+e_j, 1+e_{j+1}, \dots, 1+e_r) \\ &= \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \zeta_{\mathcal{F}}((i+e'_1, i'+e'_2, i''+e'_3)^\vee). \quad \square \end{aligned}$$

2 **Lemma 3.7.** *We have*

$$S_{k,r,i,j} = \frac{1}{2} \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \left( \begin{bmatrix} k \\ i+e'_1 \end{bmatrix} - \begin{bmatrix} k \\ i''+e'_3 \end{bmatrix} \right) \mathfrak{Z}_{\mathcal{F}}(k).$$

3 *Proof.* Using the same convention as in the statement of Lemma 2.9, we have

$$(i+e'_1, i'+e'_2, i''+e'_3)^\vee = (\{1\}^{i+e'_1-1}, 2, \{1\}^{i'+e'_2-2}, 2, \{1\}^{i''+e'_3-1}),$$

and so by Lemmas 2.9 and 3.6, we have

$$\begin{aligned} S_{k,r,i,j} &= \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \zeta_{\mathcal{F}}((i+e'_1, i'+e'_2, i''+e'_3)^\vee) \\ &= \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \zeta_{\mathcal{F}}(\{1\}^{i+e'_1-1}, 2, \{1\}^{i'+e'_2-2}, 2, \{1\}^{i''+e'_3-1}) \\ &= \frac{1}{2} \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \left( \begin{bmatrix} k \\ i+e'_1 \end{bmatrix} - \begin{bmatrix} k \\ i''+e'_3 \end{bmatrix} \right) \mathfrak{Z}_{\mathcal{F}}(k). \quad \square \end{aligned}$$

**Lemma 3.8.** *We have*

$$\begin{aligned} \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \begin{bmatrix} k \\ i+e'_1 \end{bmatrix} &= (k'+i+1) \left( \begin{bmatrix} k-1 \\ k'+i \end{bmatrix} - \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} \right) + k \left( \begin{bmatrix} k-2 \\ k'+i-1 \end{bmatrix} - \begin{bmatrix} k-2 \\ i-2 \end{bmatrix} \right), \\ \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \begin{bmatrix} k \\ i''+e'_3 \end{bmatrix} &= (k'+i''+1) \left( \begin{bmatrix} k-1 \\ k'+i'' \end{bmatrix} - \begin{bmatrix} k-1 \\ i''-1 \end{bmatrix} \right) + k \left( \begin{bmatrix} k-2 \\ k'+i''-1 \end{bmatrix} - \begin{bmatrix} k-2 \\ i''-2 \end{bmatrix} \right). \end{aligned}$$

*Proof.* By symmetry, we only need to show the first equality, which can be seen as follows:

$$\begin{aligned}
& \sum_{\substack{e'_1+e'_2+e'_3=k' \\ e'_1, e'_2, e'_3 \geq 0}} \begin{bmatrix} k \\ i+e'_1 \end{bmatrix} \\
&= \sum_{e'_1=0}^{k'} (-1)^{i+e'_1} (k' - e'_1 + 1) \binom{k}{i+e'_1} \\
&= \sum_{e'_1=0}^{k'} (-1)^{i+e'_1} ((k' + i + 1) - (i + e'_1)) \binom{k}{i+e'_1} \\
&= (k' + i + 1) \sum_{e'_1=0}^{k'} (-1)^{i+e'_1} \binom{k}{i+e'_1} - k \sum_{e'_1=0}^{k'} (-1)^{i+e'_1} \binom{k-1}{i+e'_1-1} \\
&= (k' + i + 1) \sum_{e'_1=0}^{k'} \left( (-1)^{i+e'_1} \binom{k-1}{i+e'_1} - (-1)^{i+e'_1-1} \binom{k-1}{i+e'_1-1} \right) \\
&\quad + k \sum_{e'_1=0}^{k'} \left( (-1)^{i+e'_1-1} \binom{k-2}{i+e'_1-1} - (-1)^{i+e'_1-2} \binom{k-2}{i+e'_1-2} \right) \\
&= (k' + i + 1) \left( (-1)^{k'+i} \binom{k-1}{k'+i} - (-1)^{i-1} \binom{k-1}{i-1} \right) \\
&\quad + k \left( (-1)^{k'+i-1} \binom{k-2}{k'+i-1} - (-1)^{i-2} \binom{k-2}{i-2} \right) \\
&= (k' + i + 1) \left( \begin{bmatrix} k-1 \\ k'+i \end{bmatrix} - \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} \right) + k \left( \begin{bmatrix} k-2 \\ k'+i-1 \end{bmatrix} - \begin{bmatrix} k-2 \\ i-2 \end{bmatrix} \right). \quad \square
\end{aligned}$$

1 **Lemma 3.9.** *We have*

$$S_{k,r,i,j} = \frac{1}{2} N_{k,r,i,j} \mathfrak{Z}_{\mathcal{F}}(k),$$

where  $N_{k,r,i,j}$  is an integer given by

$$\begin{aligned}
N_{k,r,i,j} &= (k' + i + 1) \left( \begin{bmatrix} k-1 \\ k'+i \end{bmatrix} - \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} \right) - (k' + i'' + 1) \left( \begin{bmatrix} k-1 \\ k'+i'' \end{bmatrix} - \begin{bmatrix} k-1 \\ i''-1 \end{bmatrix} \right) \\
&\quad + k \left( \begin{bmatrix} k-2 \\ k'+i-1 \end{bmatrix} - \begin{bmatrix} k-2 \\ i-2 \end{bmatrix} - \begin{bmatrix} k-2 \\ k'+i''-1 \end{bmatrix} + \begin{bmatrix} k-2 \\ i''-2 \end{bmatrix} \right).
\end{aligned}$$

2 *Proof.* Immediate from Lemmas 3.7 and 3.8. □

3 Lemmas 3.4 and 3.9 complete the proof of our main theorem (Theorem 3.1).

#### 4 REFERENCES

- 5 [1] M. E. Hoffman, *Quasi-symmetric functions and mod  $p$  multiple harmonic sums*, Kyushu J.  
6 Math. **69** (2015), no. 2, 345–366.  
7 [2] D. Jarossay, *Double mélange des multizêtas finis et multizêtas symétrisés*, C. R. Math. Acad.  
8 Sci. Paris **352** (2014), no. 10, 767–771.  
9 [3] M. Kaneko, *Finite multiple zeta values*, RIMS Kôkyûroku Bessatsu, to appear.  
10 [4] M. Kaneko and D. Zagier, *Finite multiple zeta values*, in preparation.



- 1 [5] H. Murahara, *A note on finite real multiple zeta values*, Kyushu J. Math. **70** (2016), no. 1,  
2 197–204.
- 3 [6] K. Oyama, *Ohno-type relation for finite multiple zeta values*, Kyushu J. Math., to appear.
- 4 [7] S. Saito, *Numerical tables of finite multiple zeta values*, RIMS Kôkyûroku Bessatsu, to ap-  
5 pear.
- 6 [8] S. Saito and N. Wakabayashi, *Sum formula for finite multiple zeta values*, J. Math. Soc.  
7 Japan **67** (2015), no. 3, 1069–1076.
- 8 [9] J. Zhao, *Multiple zeta functions, multiple polylogarithms and their special values*, Series on  
9 Number Theory and its Applications, vol. 12, World Scientific Publishing Co. Pte. Ltd.,  
10 Hackensack, NJ, 2016.

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