Knot Points of Typical Continuous Functions and Baire Category in Families of Sets of the First Class

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Abstract

Let $C(I)$ denote the Banach space of all real-valued continuous functions on the unit interval $I = [0, 1]$. We say that a *typical* function $f \in C(I)$ has a property P if the set of all $f \in C(I)$ for which the property P holds is residual in $C(I)$. We call $x \in I$ a *knot point* of $f \in C(I)$ if the Dini derivatives of f at x are appropriately positive infinite or negative infinite, and write $N(f)$ for the set of all non-knot points of $f \in C(I)$. The main theorem of the thesis characterises families $\mathcal{S}$ of subsets of I for which a typical function $f \in C(I)$ has the property that $N(f) \in \mathcal{S}$.

In order to state the main theorem, we need to define residuality of families of F_σ sets. Let $\mathcal{K}$ denote the set of all closed subsets of I, and equip it with the Hausdorff metric. Every F_σ set F can, by definition, be written as $F = \bigcup_{n=1}^{\infty} K_n$ by using an element (K_n) of the space $\mathcal{K}^{\mathbb{N}}$ of sequences of members of $\mathcal{K}$. Moreover, it is also possible to express F as $F = \bigcup_{n=1}^{\infty} K_n$ by using an element (K_n) of the space $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ of *increasing* sequences of members of $\mathcal{K}$. These observations lead us to the following two ways of defining the residuality of a family $\mathcal{F}$ of F_σ sets:

(1) the family $\mathcal{F}$ is residual if the set of all $(K_n) \in \mathcal{K}^{\mathbb{N}}$ with $\bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$ is residual in $\mathcal{K}^{\mathbb{N}}$;

(2) the family $\mathcal{F}$ is residual if the set of all $(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow}$ with $\bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

It turns out that these definitions are equivalent, and so we do not have to worry which definition to use.

Having defined the residuality, we can state the main theorem: for a family $\mathcal{S}$ of subsets of I, a typical function $f \in C(I)$ has the property that $N(f) \in \mathcal{S}$ if and only if the family of all F_σ subsets of I belonging to $\mathcal{S}$ is residual.

We use the Banach-Mazur game to prove both the main theorem and the equivalency of residuality. The usefulness of the game lies in the fact that residuality is equivalent to the existence of a winning strategy in the game.

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Contents

Notation

Notation in set theory

- $\mathbb{N} = \{1, 2, 3, \dots\}$: the set of all positive integers, excluding 0.
- $\mathbb{Z}$: the set of all integers.
- $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$: the set of all nonnegative integers, including 0.
- $\mathbb{Q}$: the set of all rational numbers.
- $\mathbb{R}$: the set of all real numbers.
- $A \subset B$, $B \supset A$: A is a subset of B, not necessarily proper.
- A^c: the complement of A.
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$: the symmetric difference of A and B.
- $A \amalg B$: the union of A and B, used only when A and B are disjoint.
- $\coprod_{\lambda \in \Lambda} A_\lambda$: the union of A_λ for $\lambda \in \Lambda$, used only when the sets A_λ are pairwise disjoint.
- $[n] = \{1, \dots, n\}$: the set of all positive integers at most n, used only when $n \in \mathbb{N}$.

Notation in topological spaces

Let X be a topological space and A a subset of X.

- $\operatorname{Int} A$: the interior of A.
- $\overline{A}$: the closure of A.

Notation in metric spaces

Let (X, d) be a metric space, and let $a \in X$, $A \subset X$, and $r > 0$.

- $B(a, r) = \{x \in X \mid d(x, a) < r\}$: the open ball around a of radius r.
- $\overline{B}(a, r) = \{x \in X \mid d(x, a) \leq r\}$: the closed ball around a of radius r.
- $B(A, r) = \bigcup_{x \in A} B(x, r)$.
- $\overline{B}(A, r) = \bigcup_{x \in A} \overline{B}(x, r)$.

Further basic notation

- $I = [0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$: the unit interval.
- $C(I)$: the Banach space consisting of all continuous functions from I to $\mathbb{R}$, with the supremum norm $\|\cdot\|$ (see Definition 1.1.1).
- $D^{\pm} f(x)$, $D_{\pm} f(x)$: the Dini derivatives of $f \in C(I)$ at $x \in I$ (see Definition 1.1.3).
- $N(f)$: the set of all points in I that are not knot points of $f \in C(I)$ (see Definition 1.1.5).

Conventions

Fonts

We shall normally use different fonts in accordance with the following rules:

- Lower case letters (a, b, $\dots$): used to denote real numbers, functions, and points of spaces.
- Upper case letters (A, B, $\dots$): used to denote sets.
- Boldface letters ($\boldsymbol{A}$, $\boldsymbol{a}$, $\dots$): used to denote sequences. A term of a sequence is denoted by the corresponding normal letter accompanied with a subscript. For example, the nth term of a sequence $\boldsymbol{x}$ is x_n.

- Calligraphic letters ($\mathcal{A}$, $\mathcal{B}$, …): used to denote families of subsets of a set.
- Calligraphic letters ($\mathscr{A}$, $\mathscr{B}$, …): used to denote more complicated objects, such as families of subsets of a more complicated set.

Superscripts

Because the complexity of the proofs given in this thesis forces us to use many indices, we shall often use superscripts as well as subscripts to denote indices rather than exponents. Although we could use brackets as in $a_n^{(m)}$ to guarantee that m is not an exponent but an index, it would sharply decrease readability with only a slight increase in clarity. We do use powers occasionally, but the meaning will always be clear from the context.

Notation defined in Chapter 3

- $\mathcal{K} = \{K \subset I \mid K \text{ is closed}\}$.
- d: the Hausdorff metric (see Definition 3.1.1).
- $\mathcal{K}^{\mathbb{N}} = \{\boldsymbol{K} = (K_n) \mid K_n \in \mathcal{K} \text{ for all } n \in \mathbb{N}\}$.
- $\overline{U}(\boldsymbol{K}, m, r) = \{\boldsymbol{L} \in \mathcal{K}^{\mathbb{N}} \mid d(K_n, L_n) \le r \text{ for all } n \in [m]\}$ for $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}$, $m \in \mathbb{N}$, and $r > 0$.
- $\mathcal{K}^{\mathbb{N}}_{\nearrow} = \{\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \mid K_1 \subset K_2 \subset \cdots\}$.
- $\overline{U}_{\nearrow}(\boldsymbol{K}, m, r) = \{\boldsymbol{L} \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid d(K_n, L_n) \le r \text{ for all } n \in [m]\}$ for $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}_{\nearrow}$, $m \in \mathbb{N}$, and $r > 0$.
- $\mathcal{F}_\sigma$: the family of all F_σ subsets of I.
- $\mathcal{K}^{\mathbb{N}}_{\mathcal{F}} = \{\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ for $\mathcal{F} \subset \mathcal{F}_\sigma$.

Notation used in Chapter 3 only

Definition 3.5.1

- $\mathcal{B} = \left\{ \overline{U}(\boldsymbol{K}, m, r) \subset \mathcal{K}^{\mathbb{N}} \;\middle|\; \begin{array}{l} \boldsymbol{K} \in \mathcal{K}^{\mathbb{N}},\ m \in \mathbb{N},\ r \in (0,1); \\ K_1, \ldots, K_m \text{ are pairwise disjoint finite sets;} \\ \text{if } x, y \in \bigcup_{n=1}^{m} K_n \text{ and } x \neq y \text{ then } |x-y| \geq 3r \end{array} \right\}.$

- $\mathcal{B}_{\nearrow} = \left\{ \overline{U}_{\nearrow}(\boldsymbol{K}, m, r) \subset \mathcal{K}^{\mathbb{N}}_{\nearrow} \;\middle|\; \begin{array}{l} \boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}_{\nearrow},\ m \in \mathbb{N},\ r \in (0,1); \\ K_m \text{ is finite;} \\ \text{if } x, y \in K_m \text{ and } x \neq y \text{ then } |x-y| \geq 3r \end{array} \right\}.$

Notation used in Chapter 4 only

Definition 4.3.1

Let $f \in C(I)$ and $a > 0$.

- $N^+(f,a)$
  $= \{x \in [0, 1-2^{-a}] \mid f(y) - f(x) \leq a(y-x) \text{ for all } y \in [x, x+2^{-a}]\}.$
- $N_+(f,a)$
  $= \{x \in [0, 1-2^{-a}] \mid f(y) - f(x) \geq -a(y-x) \text{ for all } y \in [x, x+2^{-a}]\}.$
- $N^-(f,a)$
  $= \{x \in [2^{-a}, 1] \mid f(y) - f(x) \geq a(y-x) \text{ for all } y \in [x-2^{-a}, x]\}.$
- $N_-(f,a)$
  $= \{x \in [2^{-a}, 1] \mid f(y) - f(x) \leq -a(y-x) \text{ for all } y \in [x-2^{-a}, x]\}.$
- $\hat{N}(f,a) = N^+(f,a) \cup N_-(f,a).$
- $\check{N}(f,a) = N_+(f,a) \cup N^-(f,a).$
- $N(f,a) = \hat{N}(f,a) \cup \check{N}(f,a) = N^+(f,a) \cup N_+(f,a) \cup N^-(f,a) \cup N_-(f,a).$

For $\tilde{N}(f,a)$, see Convention 4.3.2.

Definition 4.3.14

- For disjoint finite subsets $\hat{H}$ and $\check{H}$ of I and positive numbers h and w, a *bump function* of height h and width w located at $\hat{H}$ and $\check{H}$ is a function $\varphi \in C^1(I)$ with the following properties:
  - $\|\varphi\| = h$;
  - $\varphi(x) = h$ for all $x \in \hat{H}$ and $\varphi(x) = -h$ for all $x \in \check{H}$;
  - $\{x \in I \mid \varphi(x) > 0\} \subset B(\hat{H}, w)$ and $\{x \in I \mid \varphi(x) < 0\} \subset B(\check{H}, w)$.

Definition 4.3.18

- If $f \in C^1(I)$, $0 < a < b$, and $h > 0$, the positive number $\mu(f, a, b, h)$ is chosen to have the following property:

  > Suppose that φ is a bump function of height h and width $w > 0$ located at $\hat{H}$ and $\check{H}$, where $\hat{H}$ and $\check{H}$ are disjoint finite subsets of I satisfying $B(\tilde{H}, \mu) = I$. Then, setting $g = f + \varphi$, we have $\tilde{N}(g, a) \subset \tilde{N}(f, b) \cap B(\tilde{H}, w)$.

Here $B(\tilde{H}, \mu) = I$ means $B(\hat{H}, \mu) = I$ and $B(\check{H}, \mu) = I$, and $\tilde{N}(g, a) \subset \tilde{N}(f, b) \cap B(\tilde{H}, w)$ means $\hat{N}(g, a) \subset \hat{N}(f, b) \cap B(\hat{H}, w)$ and $\check{N}(g, a) \subset \check{N}(f, b) \cap B(\check{H}, w)$.

Definition 4.4.5

- $X = \{\boldsymbol{a} \in (0, \infty)^{\mathbb{N}} \mid a_1 < a_2 < \cdots \to \infty\}$.
- $Y = \{\boldsymbol{\delta} \in (0, 1)^{\mathbb{N}} \mid \delta_1 > \delta_2 > \cdots \to 0\}$.
- $Z = \{\boldsymbol{n} \in \mathbb{N}^{\mathbb{N}} \mid n_{j+1} \geq n_j + j \text{ for all } j \in \mathbb{N}\}$.
- $A_j^m(\boldsymbol{n}) = [n_j] \cup \bigcup_{i=j}^{m-1} \{n_i + 1, \ldots, n_i + j - 1\}$, where $\boldsymbol{n} \in Z$ and $j, m \in \mathbb{N}$ with $j \leq m$.
- $n_j^k = n_{j+k}$ for $\boldsymbol{n} \in Z$ and $k \in \mathbb{Z}_+$, so that $\boldsymbol{n}^k \in Z$.

- For $\boldsymbol{n} \in Z$, $\boldsymbol{\delta} \in Y$, and $k \in \mathbb{Z}_+$,

$$\mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta}) = \left\{ \boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \,\middle|\, \begin{array}{c} \displaystyle\bigcup_{n \in A_j^m(\boldsymbol{n}^k) \setminus A_j^{m-1}(\boldsymbol{n}^k)} K_n \subset \bigcup_{n \in A_{j-1}^{m-1}(\boldsymbol{n}^k)} B(K_n, \delta_m) \\ \text{whenever } 2 \leq j \leq m-1 \end{array} \right\}.$$

- $\mathscr{S}(\boldsymbol{n}, \boldsymbol{\delta}) = \bigcup_{k=0}^{\infty} \mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta}) \subset \mathcal{K}^{\mathbb{N}}$, where $\boldsymbol{n} \in Z$ and $\boldsymbol{\delta} \in Y$.

Definition 4.4.11

- For $k \in \mathbb{Z}_+$,

$$\begin{aligned} \mathscr{Y}_k = \Big\{ (\boldsymbol{K}, f, \boldsymbol{n}, \boldsymbol{\delta}, \boldsymbol{a}, \boldsymbol{b}) \in \mathcal{K}^{\mathbb{N}} \times C(I) \times Z \times Y \times X \times X \;\Big|\; & \\ & \boldsymbol{K} \in \mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta}), \\ & N(f, a_j) \subset \textstyle\bigcup_{n \in A_j^m(\boldsymbol{n}^k)} B(K_n, \delta_m) \text{ whenever } j \leq m, \\ & \textstyle\bigcup_{n \in A_j^m(\boldsymbol{n})} K_n \subset B\big(N(f, b_j), \delta_m\big) \text{ whenever } j \leq m \Big\} \end{aligned}$$

- $\mathscr{Y} = \bigcup_{k=0}^{\infty} \mathscr{Y}_k \subset \mathcal{K}^{\mathbb{N}} \times C(I) \times Z \times Y \times X \times X$.
- $\mathscr{X} = \operatorname{pr} \mathscr{Y} \subset \mathcal{K}^{\mathbb{N}} \times C(I)$, where $\operatorname{pr}\colon \mathcal{K}^{\mathbb{N}} \times C(I) \times Z \times Y \times X \times X \longrightarrow \mathcal{K}^{\mathbb{N}} \times C(I)$ is the projection.

Chapter 1

Introduction

1.1 History and background

In the study of real analysis, we often encounter an example contrary to an intuition that one may bear at first thought. More interestingly, such an example sometimes becomes a central object of study rather than just an unpleasant counterexample best to be ignored.

We can say that the history of nowhere differentiable continuous functions is one of such phenomena. Until the early nineteenth century, it was widely believed that every continuous function was differentiable at 'almost all' points. However, from around the middle of the century, several people began to discover examples of nowhere differentiable continuous functions. Furthermore, Banach [Ba] and Mazurkiewicz [Ma] independently proved in 1931 that 'most' continuous functions are nowhere differentiable. Since then, many mathematicians have been investigating properties of 'most' functions.

In the study of 'most' functions, we first have to make clear what 'most' means. Although a number of definitions have been invented, we shall use the most classical notion, upon which the above-mentioned papers by Banach and Mazurkiewicz are based.

Definition 1.1.1. We write I for the unit interval $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, and $C(I)$ for the set of all continuous functions from I to $\mathbb{R}$. It is well known that $C(I)$ is a Banach space under the supremum norm $\|\cdot\|$ defined by

$$\|f\| = \sup_{x\in I}|f(x)|$$

for $f \in C(I)$.

In a topological space, Baire category provides us with an idea of 'small' sets. Small sets in the sense of Baire category are said to be *meagre*, and sets whose complements are meagre are said to be *residual.* Properties of 'most' functions will be understood as those possessed by all functions in a residual subset of $C(I)$:

Definition 1.1.2. We say that a *typical* (or *generic*) function $f \in C(I)$ has a property P if the set of all $f \in C(I)$ with the property P is residual in $C(I)$.

As mentioned earlier, a typical function is nowhere differentiable, so its derivative cannot be considered. In place of its derivative, we shall look at its *Dini derivatives*:

Definition 1.1.3. Let $f \in C(I)$. We define

$$D^+f(x) = \limsup_{y\downarrow x}\frac{f(y)-f(x)}{y-x}, \qquad D_+f(x) = \liminf_{y\downarrow x}\frac{f(y)-f(x)}{y-x}$$

for $x \in [0,1)$, and

$$D^-f(x) = \limsup_{y\uparrow x}\frac{f(y)-f(x)}{y-x}, \qquad D_-f(x) = \liminf_{y\uparrow x}\frac{f(y)-f(x)}{y-x}$$

for $x \in (0,1]$. They are called the *Dini derivatives* of f at x.

The oldest result about the behaviour of the Dini derivatives of a typical continuous is the following theorem by Jarník [Ja]:

Theorem 1.1.4 ([Ja]). *A typical function $f \in C(I)$ has the property that*

$$D^+f(x) = D^-f(x) = \infty, \qquad D_+f(x) = D_-f(x) = -\infty$$

for almost every $x \in (0,1)$.

This theorem leads us to the following definition:

Definition 1.1.5. We say that a point $x \in I$ is a *knot point* of $f \in C(I)$ if

- $x \in (0,1)$, $D^+f(x) = D^-f(x) = \infty$, and $D_+f(x) = D_-f(x) = -\infty$; or
- $x = 0$, $D^+f(x) = \infty$, and $D_+f(x) = -\infty$; or
- $x = 1$, $D^-f(x) = \infty$, and $D_-f(x) = -\infty$.

For $f \in C(I)$, we write $N(f)$ for the set of all points in I that are *not* knot points of f.

Theorem 1.1.4 means that a typical function $f \in C(I)$ has the property that $N(f)$ is Lebesgue null, i.e. small from the measure-theoretic viewpoint. It is natural to ask in what sense of smallness it is true that a typical function has the property that $N(f)$ is small. Preiss and Zajíček answered this question in unpublished work [PZ] by giving a necessary and sufficient condition for a σ-ideal $\mathcal{I}$ (a family of 'small' sets; see Remark 2.1.3 for its definition) to satisfy that a typical function $f \in C(I)$ has the property that $N(f) \in \mathcal{I}$ (see Theorem 4.1.1 for the precise statement). The purpose of this thesis is to generalise this theorem by giving a necessary and sufficient condition for an *arbitrary* family $\mathcal{S}$ of subsets of I to satisfy that a typical function $f \in C(I)$ has the property that $N(f) \in \mathcal{S}$ (see Theorem 4.1.2 for the precise statement). The theorem has been established by Preiss and the author, and will be written in [PS].

1.2 Structure of this thesis

We first review in Chapter 2 some standard definitions and facts that will be used in subsequent chapters. A more detailed exposition and complete proofs can be found in [Ke].

Chapter 3 discusses residuality of families of F_σ sets and proves that two natural definitions of residuality are the same. The residuality will be used to state the main theorem of this thesis.

In Chapter 4 we state and prove our main theorem. Because of the high complexity with which the proof is written, the author decided to devote the last section of Chapter 4 to the outline of the proof that he believes helps the reader to understand where the technical difficulties lie, though it is logically not part of the proof.

Chapter 2

Preliminaries

2.1 Baire category

Definition 2.1.1. Let X be a topological space and A a subset of X.

(1) We say that A is *nowhere dense* if $\operatorname{Int} \overline{A} = \emptyset$.

(2) We say that A is *meagre* if A can be expressed as a countable union of nowhere dense subsets of X.

(3) We say that A is *residual* (or *comeagre*) if A^c is meagre.

Remark 2.1.2. Some people refer to meagre sets and nonmeagre sets as sets of *first category* and of *second category* respectively, which is why we call this concept *Baire category.* However, since the term *category* is used to mean a completely different notion in many areas of mathematics, we shall stick to the terms in Definition 2.1.1.

Remark 2.1.3. It is easy to see that the family $\mathcal{I}$ of all meagre subsets of a topological space X is a σ-ideal, i.e. $\mathcal{I}$ has the following properties:

(1) $\emptyset \in \mathcal{I}$;

(2) if $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$;

(3) if $A_n \in \mathcal{I}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{I}$.

Baire category gives a formulation of 'small' sets in topological spaces, but does not work very well for all topological spaces; for example, the whole space is meagre in $\mathbb{Q}$. Spaces in which the concept is meaningful are called *Baire spaces*:

Definition 2.1.4. A *Baire space* is a topological space in which no nonempty open set is meagre.

Remark 2.1.5. (1) Saying that no nonempty open set is meagre is equivalent to saying that every residual set is dense.

(2) A set A is residual if and only if there exist open dense sets U_n with $\bigcap_{n=1}^{\infty} U_n \subset A$. In a Baire space, it is also equivalent to the condition that A contains a dense G_δ set.

(3) In a nonempty Baire space, no set is both meagre and residual because the whole space is not meagre.

Complete metric spaces are important examples of Baire spaces:

Theorem 2.1.6 (Baire Category Theorem). *Every complete metric space is a Baire space.*

Proof. Let (X, d) be a complete metric space, and suppose that a nonempty open subset U of X is meagre. Then we may find nowhere dense subsets A_n of X with $U = \bigcup_{n=1}^{\infty} A_n$.

We inductively define a sequence (x_n) of points in X and a sequence (r_n) of positive numbers. Since A_1 is nowhere dense, we find that $U \setminus \overline{A}_1$ is a nonempty open set, and so there exist $x_1 \in X$ and $r_1 > 0$ such that $\overline{B}(x_1, 2r_1) \subset U \setminus \overline{A}_1$. Suppose that x_n and r_n have been defined. Since A_{n+1} is nowhere dense, we find that $B(x_n, r_n) \setminus \overline{A}_{n+1}$ is a nonempty open set, and so there exist $x_{n+1} \in X$ and $r_{n+1} > 0$ such that $\overline{B}(x_{n+1}, 2r_{n+1}) \subset B(x_n, r_n) \setminus \overline{A}_{n+1}$ and $r_{n+1} < r_n/2$.

Note that $d(x_n, x_{n+1}) < r_n$ for every $n \in \mathbb{N}$. Therefore, if $m < n$, then

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) < \sum_{k=m}^{n-1} r_k \leq \sum_{k=m}^{n-1} 2^{-(k-m)} r_m < 2r_m.$$

It follows that (x_n) is a Cauchy sequence, and so it is convergent, say to x. The inequality shown above implies that $d(x_m, x) \leq 2r_m$ for all $m \in \mathbb{N}$. Hence the point x belongs to U but does not belong to any A_n, which contradicts the choice of A_n. ∎

2.2 Banach-Mazur game

Definition 2.2.1 (Banach-Mazur game). Let X be a topological space, S a subset of X, and $\mathcal{A}$ a family of subsets of X. Suppose that every set in $\mathcal{A}$ has nonempty interior and that every nonempty open subset of X contains a set in $\mathcal{A}$. The $(X, S, \mathcal{A})$-*Banach-Mazur game* is described as follows. Two players, called Player I and Player II, alternately choose a set in $\mathcal{A}$ with the restriction that each player must choose a subset of the set chosen by the other player in the previous turn. Player II will win if the intersection of all the sets chosen by the players is contained in S; otherwise Player I will win.

Remark 2.2.2. Let A_n and B_n be the sets chosen in the nth round by Players I and II respectively. The rule demands that

$$A_1 \supset B_1 \supset A_2 \supset B_2 \supset \cdots,$$

which implies that the intersection we look at is the same as both $\bigcap_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} B_n$.

Example 2.2.3. The two conditions imposed on $\mathcal{A}$ in Definition 2.2.1 might seem slightly intricate. We should first note that they are fulfilled if $\mathcal{A}$ is the family of all nonempty open subsets of X. In fact, we may assume that $\mathcal{A}$ is this family when we consider which player has a winning strategy (Theorem 2.2.4). However, when we play the game in concrete spaces, it is often technically convenient to take other families as $\mathcal{A}$, which is why we allowed other families in Definition 2.2.1. We give some examples of families satisfying the conditions:

- $\mathcal{A}$ is an open base for X such that $\emptyset \notin \mathcal{A}$;

- X is a metric space and $\mathcal{A}$ is the family of all open balls;
- X is a metric space and $\mathcal{A}$ is the family of all closed balls;
- X is a metric space, D is a dense subset of X, and $\mathcal{A}$ is the family of all open balls whose centres belong to D.

There is an easy criterion for deciding whether Player II has a winning strategy in the Banach-Mazur game:

Theorem 2.2.4 ([Ox, Theorem 1]). *The $(X, S, \mathcal{A})$-Banach-Mazur game admits a winning strategy for Player II if and only if S is residual in X.*

Proof. Firstly, assuming that S is residual in X, we shall give a winning strategy for Player II. We may take open dense subsets U_n of X with $\bigcap_{n=1}^{\infty} U_n \subset S$. Let $A_n \in \mathcal{A}$ be the set chosen by Player I in the nth round. Since $\operatorname{Int} A_n$ is a nonempty open set, its intersection with U_n is also a nonempty open set. It follows that there exists $B_n \in \mathcal{A}$ with $B_n \subset \operatorname{Int} A_n \cap U_n$. Player II will choose B_n as her nth move. Note that this move is legal because $B_n \subset \operatorname{Int} A_n \subset A_n$. If Player II adopts this strategy, then

$$\bigcap_{n=1}^{\infty} B_n \subset \bigcap_{n=1}^{\infty} U_n \subset S,$$

which implies that Player II wins.

Conversely, suppose that Player II has a winning strategy. For each $n \in \mathbb{N}$, let $\mathscr{X}_n$ denote the set of all $(A_1, B_1, \ldots, A_n, B_n) \in \mathcal{A}^{2n}$ such that, for every $j \in [n]$, the strategy tells Player II to reply B_j when the first j moves of Player I are $A_1, \ldots, A_j$.

We shall inductively construct $\mathscr{Y}_n$ and $\mathscr{Z}_n$ with $\mathscr{Y}_n \subset \mathscr{Z}_n \subset \mathscr{X}_n$ using Zorn's lemma. Firstly, we set $\mathscr{Z}_1 = \mathscr{X}_1$ and take a maximal subset $\mathscr{Y}_1$ of $\mathscr{Z}_1$ such that if (A_1, B_1) and (A_1', B_1') are distinct elements of $\mathscr{Y}_1$, then $\operatorname{Int} B_1 \cap \operatorname{Int} B_1' = \emptyset$. When $\mathscr{Y}_n$ has been defined, we set

$$\mathscr{Z}_{n+1} = \{(A_1, B_1, \ldots, A_{n+1}, B_{n+1}) \in \mathscr{X}_{n+1} \mid (A_1, B_1, \ldots, A_n, B_n) \in \mathscr{Y}_n\}$$

and take a maximal subset $\mathscr{Y}_{n+1}$ of $\mathscr{Z}_{n+1}$ such that if $(A_1, B_1, \ldots, A_{n+1}, B_{n+1})$ and $(A'_1, B'_1, \ldots, A'_{n+1}, B'_{n+1})$ are distinct elements of $\mathscr{Y}_{n+1}$, then $\operatorname{Int} B_{n+1} \cap \operatorname{Int} B'_{n+1} = \emptyset$. Note that for every $(A_1, B_1, \ldots, A_n, B_n) \in \mathscr{Z}_n$ there exists $(A'_1, B'_1, \ldots, A'_n, B'_n) \in \mathscr{Y}_n$ with $\operatorname{Int} B_n \cap \operatorname{Int} B'_n \neq \emptyset$, because if such an element does not exist, then the maximality of $\mathscr{Y}_n$ implies that $(A_1, B_1, \ldots, A_n, B_n)$ belongs to $\mathscr{Y}_n$, in which case, setting $(A'_1, B'_1, \ldots, A'_n, B'_n) = (A_1, B_1, \ldots, A_n, B_n)$, we have $\operatorname{Int} B_n \cap \operatorname{Int} B'_n = \operatorname{Int} B_n \neq \emptyset$, a contradiction.

Set

$$U_n = \coprod_{(A_1, B_1, \ldots, A_n, B_n) \in \mathscr{Y}_n} \operatorname{Int} B_n.$$

for each $n \in \mathbb{N}$. Obviously U_n is open for every $n \in \mathbb{N}$. We shall inductively show that U_n is dense for every $n \in \mathbb{N}$. Let U be an arbitrary nonempty open subset of X. We need to prove that $U \cap U_n \neq \emptyset$ for every $n \in \mathbb{N}$. We may take $A_1 \in \mathcal{A}$ contained in U and $B_1 \in \mathcal{A}$ with $(A_1, B_1) \in \mathscr{X}_1 = \mathscr{Z}_1$. Then there exists $(A'_1, B'_1) \in \mathscr{Y}_1$ with $\operatorname{Int} B_1 \cap \operatorname{Int} B'_1 \neq \emptyset$, which implies that

$$U \cap U_1 \supset B_1 \cap \operatorname{Int} B'_1 \neq \emptyset.$$

Assume that we have proved that $U \cap U_n \neq \emptyset$. It means that there exists $(A_1, B_1, \ldots, A_n, B_n) \in \mathscr{Y}_n$ such that $U \cap \operatorname{Int} B_n \neq \emptyset$. We may take $A_{n+1} \in \mathcal{A}$ contained in $U \cap \operatorname{Int} B_n$ and $B_{n+1} \in \mathcal{A}$ with $(A_1, B_1, \ldots, A_{n+1}, B_{n+1}) \in \mathscr{X}_{n+1}$. Since $(A_1, B_1, \ldots, A_{n+1}, B_{n+1}) \in \mathscr{Z}_{n+1}$, there exists $(A'_1, B'_1, \ldots, A'_{n+1}, B'_{n+1}) \in \mathscr{Y}_{n+1}$ with $\operatorname{Int} B_{n+1} \cap \operatorname{Int} B'_{n+1} \neq \emptyset$, which implies that

$$U \cap U_{n+1} \supset B_{n+1} \cap \operatorname{Int} B'_{n+1} \neq \emptyset.$$

Having shown that U_n is open dense for every $n \in \mathbb{N}$, we only need to prove that $\bigcap_{n=1}^{\infty} U_n \subset S$. Let $x \in \bigcap_{n=1}^{\infty} U_n$. For each $n \in \mathbb{N}$, there exists a unique $(A_1^n, B_1^n, \ldots, A_n^n, B_n^n) \in \mathscr{Y}_n$ such that $x \in \operatorname{Int} B_n^n$. For every $n \in \mathbb{N}$, since

$$x \in \operatorname{Int} B_{n+1}^{n+1} \subset \operatorname{Int} B_n^{n+1}$$

and $(A_1^{n+1}, B_1^{n+1}, \ldots, A_n^{n+1}, B_n^{n+1}) \in \mathscr{Y}_n$, the uniqueness shows that

$$(A_1^n, B_1^n, \ldots, A_n^n, B_n^n) = (A_1^{n+1}, B_1^{n+1}, \ldots, A_n^{n+1}, B_n^{n+1}).$$

It follows that neither A_j^n nor B_j^n depends on n, so we have found a sequence $(A_1, B_1, A_2, B_2, \dots) \in \mathcal{A}^{\mathbb{N}}$ such that $(A_1, B_1, \dots, A_n, B_n) \in \mathscr{Y}_n$ and $x \in \operatorname{Int} B_n$ for all $n \in \mathbb{N}$. Because $(A_1, B_1, \dots, A_n, B_n) \in \mathscr{X}_n$ for all $n \in \mathbb{N}$ and Player II is adopting a winning strategy, we find that $\bigcap_{n=1}^{\infty} B_n \subset S$, which implies that

$$x \in \bigcap_{n=1}^{\infty} \operatorname{Int} B_n \subset \bigcap_{n=1}^{\infty} B_n \subset S.$$

∎

Remark 2.2.5. Later in this thesis, we shall show residuality by constructing a winning strategy of a Banach-Mazur game. As the proof shows, it is much more difficult to prove that the existence of a winning strategy implies the residuality than its converse. It means that constructing a winning strategy is easier than verifying residuality directly.

2.3 Analytic sets and Baire property

Definition 2.3.1. A *Polish space* is a topological space that is second countable and completely metrisable.

Example 2.3.2. Among the easiest examples of Polish spaces are $\mathbb{R}$, I, and $\mathbb{N}$. The open intervals $(0, 1)$ and $(0, \infty)$, where the Euclidean metric is not complete, are also Polish because they are homeomorphic to $\mathbb{R}$. To observe that $C(I)$ is Polish, we need to note that the polynomial functions with rational coefficients form a dense subset of $C(I)$.

Every compact metric space is second countable and therefore is Polish.

Proposition 2.3.3 ([Ke, Proposition 3.3 and Theorem 3.11]). (1) *The product of countably many Polish spaces is always Polish.*

(2) *Every G_δ subset of a Polish space is Polish.*

Definition 2.3.4. Let X be a Polish space. A subset A of X is said to be *analytic* if there exist a Polish space Y and a Borel subset B of $X \times Y$ such that $A = \operatorname{pr} B$, where pr denotes the projection from $X \times Y$ to X.

Proposition 2.3.5. (1) *Every Borel subset of a Polish space is analytic.*

(2) *The family of all analytic subsets of a Polish space is closed under taking countable unions and countable intersections.*

(3) *If X and Y are Polish spaces and $f\colon X \longrightarrow Y$ is continuous, then $f(A)$ is analytic for every analytic subset A of X.*

Proof. If X is a Polish space and B is a Borel subset of X, then $B \times X$ is a Borel subset of $X \times X$ whose projection to the first coordinate is B, so B is analytic. This proves (1); see Proposition 14.4 of [Ke] for (2) and (3) (we also need Exercise 14.3 because the definition of analytic sets is slightly different in [Ke]). ∎

Definition 2.3.6. Let X be a topological space. A subset A of X is said to have the *Baire property* if there exist an open subset U of X and a meagre subset M of X such that $A = U \triangle M$.

Proposition 2.3.7 ([Ke, Proposition 8.22]). *Let X be a topological space. Then the family of all subsets of X with the Baire property is a σ-algebra on X.*

Theorem 2.3.8 ([Ke, Theorem 21.6]). *Let X be a Polish space. Then every analytic subset of X has the Baire property.*

Chapter 3

Baire category in families of sets of the first class

3.1 Hausdorff metric

3.1.1 The space $\mathcal{K}$

Definition 3.1.1. We write $\mathcal{K}$ for the set of all closed (or equivalently compact) subsets of I. The *Hausdorff metric* d on $\mathcal{K}$ is defined by

$$d(K, L) = \inf\{r > 0 \mid B(K, r) \supset L,\ B(L, r) \supset K\}$$

if neither K nor L is empty and by

$$d(K, L) = \begin{cases} 1 & \text{if exactly one of } K \text{ and } L \text{ is empty;} \\ 0 & \text{if both } K \text{ and } L \text{ are empty.} \end{cases}$$

Remark 3.1.2. For $K, L \in \mathcal{K}$ and $r \in (0, 1)$,

- $d(K, L) < r$ if and only if $K \subset B(L, r)$ and $L \subset B(K, r)$;
- $d(K, L) \leq r$ if and only if $K \subset \overline{B}(L, r)$ and $L \subset \overline{B}(K, r)$

even when either K or L is empty.

Proposition 3.1.3. *The space $\mathcal{K}$ equipped with the Hausdorff metric is compact and therefore is Polish.*

Proof. See Theorem 4.26 of [Ke]. ∎

Proposition 3.1.4 ([Ke, Exercise 4.29]). (1) *The set $\{(x, K) \in I \times \mathcal{K} \mid x \in K\}$ is closed in $I \times \mathcal{K}$.*

(2) *The set $\{(K, L) \in \mathcal{K}^2 \mid K \subset L\}$ is closed in $\mathcal{K}^2$.*

(3) *The map $\mathcal{K}^n \longrightarrow \mathcal{K};\ (K_1, \dots, K_n) \longmapsto \bigcup_{j=1}^n K_j$ is continuous for every $n \in \mathbb{N}$.*

3.1.2 The product space $\mathcal{K}^{\mathbb{N}}$

Definition 3.1.5. We denote by $\mathcal{K}^{\mathbb{N}}$ the set of all sequences of members of $\mathcal{K}$, and equip it with the product topology.

Proposition 3.1.6. *The space $\mathcal{K}^{\mathbb{N}}$ is a compact metrisable space.*

Proof. Use Proposition 3.1.3 and invoke the fact that compactness and metrisability are closed under taking countable products. ∎

Definition 3.1.7. For $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}$, $m \in \mathbb{N}$, and $r > 0$, we set

$$\overline{U}(\boldsymbol{K}, m, r) = \{\boldsymbol{L} \in \mathcal{K}^{\mathbb{N}} \mid d(K_n, L_n) \le r \text{ for all } n \in [m]\}.$$

Remark 3.1.8. Observe that $\overline{U}(\boldsymbol{K}, m, r)$ is a closed subset of $\mathcal{K}^{\mathbb{N}}$ with nonempty interior for every $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}$, $m \in \mathbb{N}$, and $r > 0$. It follows from the definition of the product topology that for every $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}$ and every open neighbourhood $\mathscr{U}$ of $\boldsymbol{K}$, there exist $m \in \mathbb{N}$ and $r > 0$ satisfying $\overline{U}(\boldsymbol{K}, m, r) \subset \mathscr{U}$.

3.1.3 The subspace $\mathcal{K}^{\mathbb{N}}_{\nearrow}$

Definition 3.1.9. We denote by $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ the subset of $\mathcal{K}^{\mathbb{N}}$ consisting of all increasing sequences:

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} = \{\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \mid K_1 \subset K_2 \subset \cdots\},$$

and equip it with the relative topology.

Proposition 3.1.10. *The subset $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ is closed in $\mathcal{K}^{\mathbb{N}}$, and so it is a compact metrisable space.*

Proof. Observe that

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} = \bigcap_{n=1}^{\infty} \{\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \mid K_n \subset K_{n+1}\}$$

and use Proposition 3.1.4 (2). ∎

Definition 3.1.11. For $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}_{\nearrow}$, $m \in \mathbb{N}$, and $r > 0$, we set

$$\overline{U}_{\nearrow}(\boldsymbol{K}, m, r) = \{\boldsymbol{L} \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid d(K_n, L_n) \leq r \text{ for all } n \in [m]\}.$$

Remark 3.1.12. Observe that $\overline{U}_{\nearrow}(\boldsymbol{K}, m, r)$ is a closed subset of $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ with nonempty interior for every $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}_{\nearrow}$, $m \in \mathbb{N}$, and $r > 0$. It follows from Remark 3.1.8 and the definition of the relative topology that for every $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}_{\nearrow}$ and every open neighbourhood $\mathscr{U}$ of $\boldsymbol{K}$, there exist $m \in \mathbb{N}$ and $r > 0$ satisfying $\overline{U}_{\nearrow}(\boldsymbol{K}, m, r) \subset \mathscr{U}$.

3.2 Residuality of families of F_σ sets

Definition 3.2.1. We write $\mathcal{F}_\sigma$ for the family of all F_σ subsets of I.

Definition 3.2.2. For a subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$, we put

$$\mathcal{K}^{\mathbb{N}}_{\mathcal{F}} = \left\{\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \,\middle|\, \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\right\}.$$

We say that $\mathcal{F}$ is *residual* if $\mathcal{K}^{\mathbb{N}}_{\mathcal{F}}$ is residual in $\mathcal{K}^{\mathbb{N}}$ and that $\mathcal{F}$ is $\nearrow$*-residual* if $\mathcal{K}^{\mathbb{N}}_{\mathcal{F}} \cap \mathcal{K}^{\mathbb{N}}_{\nearrow}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

The following is our main theorem in this chapter and asserts that these two notions of residuality are the same:

Theorem 3.2.3 (Main Theorem in Chapter 3). *A subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$ is residual if and only if it is $\nearrow$-residual.*

The proof of this theorem will be given in Section 3.5.

Remark 3.2.4. Theorem 3.2.3 remains true if we replace I by a compact dense-in-itself metric space; the same proof works.

3.3 Residuality of σ-ideals of F_σ sets

Lemma 3.3.1. *Let X and Y be topological spaces and suppose that X is second countable. If A is a residual subset of $X \times Y$, then*

$$\big\{y \in Y \;\big|\; \{x \in X \mid (x, y) \in A\} \text{ is residual}\big\}$$

is residual.

Proof. We may assume that X is nonempty, and we take a countable base $\{U_n\}_{n\in\mathbb{N}}$ for X such that $U_n \neq \emptyset$ for all $n \in \mathbb{N}$. Since A is residual, we may take open dense subsets G_m of $X \times Y$ such that $\bigcap_{m=1}^{\infty} G_m \subset A$. For $m, n \in \mathbb{N}$, write V_{mn} for the projection of $G_m \cap (U_n \times Y)$ to Y. Every V_{mn} is open because the projection is an open map. Moreover, every V_{mn} is dense because if O is a nonempty open subset of Y, then the nonempty open set $U_n \times O$ meets the dense set G_m, which means that $V_{mn} \cap O \neq \emptyset$. Therefore $\bigcap_{m,n=1}^{\infty} V_{mn}$ is residual.

We now only need to show that if $y \in \bigcap_{m,n=1}^{\infty} V_{mn}$, then $\{x \in X \mid (x, y) \in A\}$ is residual. The set $\{x \in X \mid (x, y) \in G_m\}$ is open because so is G_m; it is dense because it meets every U_n by the assumption on y. Hence the result follows from the observation that

$$\bigcap_{m=1}^{\infty} \{x \in X \mid (x, y) \in G_m\} \subset \{x \in X \mid (x, y) \in A\}.$$

∎

Remark 3.3.2. The foregoing lemma is part of the Kuratowski-Ulam theorem; see Theorem 8.41 of [Ke] for the whole theorem.

Lemma 3.3.3. *Let X be a second countable topological space and Y a nonempty Baire space. Then a subset A of X is residual if and only if $A \times Y$ is residual in $X \times Y$.*

Proof. Suppose first that $A \times Y$ is residual. Then since Y is a nonempty Baire space, Lemma 3.3.1 shows that $\{x \in X \mid (x, y) \in A \times Y\}$ is residual for some $y \in Y$. It means that A is residual.

Conversely, suppose that A is residual. Take open dense subsets G_n of X such that $\bigcap_{n=1}^{\infty} G_n \subset A$. Then $G_n \times Y$ is open dense and $\bigcap_{n=1}^{\infty}(G_n \times Y) \subset A \times Y$, from which we may conclude that $A \times Y$ is residual. ∎

Proposition 3.3.4. *If $\mathcal{I}$ is a σ-ideal on I, then $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$ if and only if $\mathcal{I} \cap \mathcal{F}_\sigma$ is residual in $\mathcal{F}_\sigma$.*

Proof. Since $\mathcal{I}$ is a σ-ideal, we have

$$\begin{aligned} \left\{ \boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{I} \right\} &= \{\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \mid K_n \in \mathcal{I} \text{ for every } n \in \mathbb{N}\} \\ &= \bigcap_{n=1}^{\infty} \big(\underbrace{\mathcal{K} \times \cdots \times \mathcal{K}}_{n-1 \text{ times}} \times (\mathcal{I} \cap \mathcal{K}) \times \mathcal{K} \times \mathcal{K} \times \cdots \big). \end{aligned}$$

Therefore $\mathcal{I} \cap \mathcal{F}_\sigma$ is residual in $\mathcal{F}_\sigma$ if and only if $(\mathcal{I} \cap \mathcal{K}) \times \mathcal{K} \times \mathcal{K} \times \cdots$ is residual in $\mathcal{K}^{\mathbb{N}}$. Lemma 3.3.3 shows that the latter condition is equivalent to $\mathcal{I} \cap \mathcal{K}$ being residual in $\mathcal{K}$. This proves the required equivalency. ∎

3.4 Universal sets

Definition 3.4.1. Let X be a Polish space. We say that a subset A of $I \times X$ is X-*universal* for $\mathcal{F}_\sigma$ if it has the following properties:

- A is an F_σ subset of $I \times X$;

- a subset F of I is F_σ if and only if $F = \{t \in I \mid (t, x) \in A\}$ for some $x \in X$.

Remark 3.4.2. For every uncountable Polish space X, there exists an X-universal set for $\mathcal{F}_\sigma$ (see [Ke, Exercise 22.6]).

If A is X-universal for $\mathcal{F}_\sigma$, then it is natural to define residuality of families of F_σ sets by declaring that $\mathcal{F} \subset \mathcal{F}_\sigma$ is residual if

$$\bigl\{x \in X \bigm| \{t \in I \mid (t, x) \in A\} \in \mathcal{F}\bigr\}$$

is residual. Observe from the following proposition that our definitions of residuality and $\nearrow$-residuality (Definition 3.2.2) are special cases of this definition of residuality:

Proposition 3.4.3. *The sets*

$$\left\{(\boldsymbol{K}, x) \in \mathcal{K}^{\mathbb{N}} \times I \,\middle|\, x \in \bigcup_{n=1}^{\infty} K_n\right\}, \quad \left\{(\boldsymbol{K}, x) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \times I \,\middle|\, x \in \bigcup_{n=1}^{\infty} K_n\right\}$$

are $\mathcal{K}^{\mathbb{N}}$- and $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-universal for $\mathcal{F}_\sigma$ respectively.

Proof. We shall prove the $\mathcal{K}^{\mathbb{N}}$-universality of the former set only; the same reasoning applies to the latter set as well. Denote the set by $\mathscr{A}$. Since

$$\mathscr{A} = \bigcup_{n=1}^{\infty} \{(\boldsymbol{K}, x) \in \mathcal{K}^{\mathbb{N}} \times I \mid x \in K_n\}$$

and each set $\{(\boldsymbol{K}, x) \in \mathcal{K}^{\mathbb{N}} \times I \mid x \in K_n\}$ is the inverse image of the closed set $\{(K, x) \in \mathcal{K} \times I \mid x \in K\}$ (Proposition 3.1.4 (1)) under the projection $\mathcal{K}^{\mathbb{N}} \times I \longrightarrow \mathcal{K} \times I$; $(\boldsymbol{K}, x) \longmapsto (K_n, x)$, we find that $\mathscr{A}$ is F_σ. The other requirement for $\mathscr{A}$ to be universal follows from the definition of F_σ sets. ∎

Therefore Theorem 3.2.3 means that these two universal sets yield the same residuality. However, as the following two propositions show, it is not true that all universal sets give rise to the same residuality:

Proposition 3.4.4 ([Ke, Exercise 23.21]). *The set*

$$\left\{(\boldsymbol{f}, x) \in C(I)^{\mathbb{N}} \times I \;\middle|\; \inf_{n\in\mathbb{N}} f_n(x) > 0\right\}$$

is $C(I)^{\mathbb{N}}$-universal for $\mathcal{F}_\sigma$.

Proposition 3.4.5. *Let $\mathcal{F}$ be a subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$. Then*

$$\left\{\boldsymbol{f} \in C(I)^{\mathbb{N}} \;\middle|\; \left\{x \in I \;\middle|\; \inf_{n\in\mathbb{N}} f_n(x) > 0\right\} \in \mathcal{F}\right\}$$

is residual if and only if $\emptyset \in \mathcal{F}$.

Proof. Define

$$\mathscr{A} = \left\{\boldsymbol{f} \in C(I)^{\mathbb{N}} \;\middle|\; \inf_{n\in\mathbb{N}} f_n(x) = -\infty \text{ for all } x \in I\right\}.$$

We first show that the residuality of $\mathscr{A}$ implies the proposition. So suppose for the moment that $\mathscr{A}$ is residual, and write

$$\mathscr{A}_{\mathcal{F}} = \left\{\boldsymbol{f} \in C(I)^{\mathbb{N}} \;\middle|\; \left\{x \in I \;\middle|\; \inf_{n\in\mathbb{N}} f_n(x) > 0\right\} \in \mathcal{F}\right\}$$

for $\mathcal{F} \subset \mathcal{F}_\sigma$. Note that $\{x \in I \mid \inf_{n\in\mathbb{N}} f_n(x) > 0\} = \emptyset$ for every $\boldsymbol{f} \in \mathscr{A}$. Therefore, if $\mathcal{F} \subset \mathcal{F}_\sigma$ has $\emptyset$ as its element, then $\mathscr{A}_{\mathcal{F}}$ contains $\mathscr{A}$ and so it is residual. Conversely, if $\mathscr{A}_{\mathcal{F}}$ is residual, then $\mathscr{A}_{\mathcal{F}} \cap \mathscr{A}$ is nonempty; we take an element $\boldsymbol{f}$ of $\mathscr{A}_{\mathcal{F}} \cap \mathscr{A}$ to conclude that $\emptyset = \{x \in I \mid \inf_{n\in\mathbb{N}} f_n(x) > 0\} \in \mathcal{F}$.

We now turn to the proof that $\mathscr{A}$ is residual. For $t \in \mathbb{R}$, let $\mathscr{A}_t$ denote the set of all $\boldsymbol{f} \in C(I)^{\mathbb{N}}$ such that $\max_{x\in I} f_n(x) < t$ for some $n \in \mathbb{N}$. Since $\mathscr{A} \supset \bigcap_{t\in\mathbb{Q}} \mathscr{A}_t$, it suffices to show that each $\mathscr{A}_t$ is open dense.

The density of $\mathscr{A}_t$ follows from the fact that it contains all sequences in $C(I)^{\mathbb{N}}$ of whose terms at least one is the constant function $t - 1$. To prove that $\mathscr{A}_t$ is open, it is enough to show that for every $n \in \mathbb{N}$ the set of all $\boldsymbol{f} \in C(I)^{\mathbb{N}}$ satisfying $\max_{x\in I} f_n(x) < t$ is open. This follows from the observation that this set is the inverse image of the open interval $(-\infty, t)$ under the composite of the projection $C(I)^{\mathbb{N}} \longrightarrow C(I)$; $\boldsymbol{f} \longmapsto f_n$ and the continuous function $C(I) \longrightarrow \mathbb{R}$; $f \longmapsto \max_{x\in I} f(x)$, whose continuity follows from the inequality

$$\left|\max_{x\in I} f(x) - \max_{x\in I} g(x)\right| \leq \|f - g\|$$

for $f, g \in C(I)$. ∎

3.5 Proof of Theorem 3.2.3

We shall prove Theorem 3.2.3 in this section, throughout which we fix an arbitrary subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$. The terms, symbols, and conventions introduced in this section are valid within this section only.

3.5.1 Games we consider here

Definition 3.5.1. Let $\mathcal{B}$ denote the family of all subsets of $\mathcal{K}^{\mathbb{N}}$ that can be written as $\overline{U}(\boldsymbol{K}, m, r)$ for some $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}$, $m \in \mathbb{N}$, and $r \in (0,1)$ such that K_1, $\dots$, K_m are pairwise disjoint finite sets and any two distinct points in $\bigcup_{n=1}^{m} K_n$ have distance at least $3r$.

Let $\mathcal{B}_{\nearrow}$ denote the family of all subsets of $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ that can be written as $\overline{U}_{\nearrow}(\boldsymbol{K}, m, r)$ for some $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}_{\nearrow}$, $m \in \mathbb{N}$, and $r \in (0,1)$ such that K_m is finite and any two distinct points in K_m have distance at least $3r$.

Remark 3.5.2. Whenever we write $\overline{U}(\boldsymbol{K}, m, r) \in \mathcal{B}$ or $\overline{U}_{\nearrow}(\boldsymbol{K}, m, r) \in \mathcal{B}_{\nearrow}$, we understand that $\boldsymbol{K}$, m, and r satisfy the conditions in Definition 3.5.1.

Remark 3.5.3. Let $\overline{U}(\boldsymbol{K}, a, r), \overline{U}(\boldsymbol{L}, b, s) \in \mathcal{B}$. It is easy to observe that if $\boldsymbol{K} = \boldsymbol{L}$, $a \geq b$, and $r \leq s$, then $\overline{U}(\boldsymbol{K}, a, r) \subset \overline{U}(\boldsymbol{L}, b, s)$. The same is true for $\mathcal{B}_{\nearrow}$. Note that $\overline{U}(\boldsymbol{K}, a, r) \subset \overline{U}(\boldsymbol{L}, b, s)$ does not imply $r \leq s$; for example, consider the case where $K_1 = \{0\}$, $L_1 = \{0.1\}$, $a = b = 1$, $r = 0.2$, and $s = 0.1$.

Definition 3.5.4. By the *game* we mean the $(\mathcal{K}^{\mathbb{N}}, \mathcal{K}^{\mathbb{N}}_{\mathcal{F}}, \mathcal{B})$-Banach-Mazur game, and by the $\nearrow$*-game* we mean the $(\mathcal{K}^{\mathbb{N}}_{\nearrow}, \mathcal{K}^{\mathbb{N}}_{\mathcal{F}} \cap \mathcal{K}^{\mathbb{N}}_{\nearrow}, \mathcal{B}_{\nearrow})$-Banach-Mazur game.

Lemma 3.5.5. *Theorem* 3.2.3 *is equivalent to the assertion that the game admits a winning strategy for Player II if and only if the* $\nearrow$*-game does.*

Proof. Immediate from Theorem 2.2.4. ∎

3.5.2 Outline of the proof

We outline the proof that if the $\nearrow$-game admits a winning strategy for Player II, then so does the game. Figure 3.1 illustrates this, and Figure 3.2 helps us to imagine the outline of the proof of the other implication.

Suppose that Player I has chosen $\overline{U}(\boldsymbol{K}^1, a^1, r^1) \in \mathcal{B}$ as his first move. Player II transfers it to a certain set, say $\overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}^1, \tilde{a}^1, \tilde{r}^1) \in \mathcal{B}_{\nearrow}$, in the $\nearrow$-game. Then the winning strategy for the $\nearrow$-game tells Player II to take a set $\overline{U}_{\nearrow}(\boldsymbol{L}^1, b^1, s^1) \in \mathcal{B}_{\nearrow}$, which she transfers to get her real reply $\overline{U}(\tilde{\boldsymbol{L}}^1, \tilde{b}^1, \tilde{s}^1) \in \mathcal{B}$ in the game. In a similar manner, after Player I replies $\overline{U}(\boldsymbol{K}^2, a^2, r^2) \in \mathcal{B}$, Player II obtains $\overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}^2, \tilde{a}^2, \tilde{r}^2) \in \mathcal{B}_{\nearrow}$, $\overline{U}_{\nearrow}(\boldsymbol{L}^2, b^2, s^2) \in \mathcal{B}_{\nearrow}$, and $\overline{U}(\tilde{\boldsymbol{L}}^2, \tilde{b}^2, \tilde{s}^2) \in \mathcal{B}$. Player II continues this strategy.

Since $\mathcal{K}^{\mathbb{N}}$ and $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ are both compact, in either game the intersection of the closed sets chosen by the players is nonempty. By modifying the winning strategy for the $\nearrow$-game, we will assume that $b^m \geq \tilde{a}^m$ and $s^m \leq \tilde{r}^m$ for all $m \in \mathbb{N}$ and that $\lim_{m\to\infty} b^m = \infty$ and $\lim_{m\to\infty} s^m = 0$, so that the intersection in the $\nearrow$-game will be a singleton. Furthermore, since the transfers will be executed in such a way that $\tilde{b}^m \geq b^m$ and $\tilde{s}^m \leq s^m$ for all $m \in \mathbb{N}$, the intersection in the game will also be a singleton. It means that we may write

$$\bigcap_{m=1}^{\infty} \overline{U}(\boldsymbol{K}^m, a^m, r^m) = \bigcap_{m=1}^{\infty} \overline{U}(\tilde{\boldsymbol{L}}^m, \tilde{b}^m, \tilde{s}^m) = \{\boldsymbol{P}\},$$
$$\bigcap_{m=1}^{\infty} \overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}^m, \tilde{a}^m, \tilde{r}^m) = \bigcap_{m=1}^{\infty} \overline{U}_{\nearrow}(\boldsymbol{L}^m, b^m, s^m) = \{\boldsymbol{Q}\}.$$

Observe that

$$\lim_{m\to\infty} K_n^m = \lim_{m\to\infty} \tilde{L}_n^m = P_n, \qquad \lim_{m\to\infty} \tilde{K}_n^m = \lim_{m\to\infty} L_n^m = Q_n$$

for all $n \in \mathbb{N}$.

In order to prove that this strategy for Player II in the game is winning, we will need to check that $\boldsymbol{P} \in \mathcal{K}^{\mathbb{N}}_{\mathcal{F}}$. Since Player II follows the winning strategy in the $\nearrow$-game, we know that $\boldsymbol{Q} \in \mathcal{K}^{\mathbb{N}}_{\mathcal{F}} \cap \mathcal{K}^{\mathbb{N}}_{\nearrow}$. Therefore it will be enough to show that $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$.

$$
\begin{array}{lcccl}
 & \text{game} & & \nearrow\text{-game} & \\
\text{Player I:} & \overline{U}(\boldsymbol{K}^1, a^1, r^1) & \longrightarrow & \overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}^1, \tilde{a}^1, \tilde{r}^1) & \\
 & & & & \text{strategy} \\
\text{Player II:} & \overline{U}(\tilde{\boldsymbol{L}}^1, \tilde{b}^1, \tilde{s}^1) & \longleftarrow & \overline{U}_{\nearrow}(\boldsymbol{L}^1, b^1, s^1) & \\
\text{Player I:} & \overline{U}(\boldsymbol{K}^2, a^2, r^2) & \longrightarrow & \overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}^2, \tilde{a}^2, \tilde{r}^2) & \\
 & & & & \text{strategy} \\
\text{Player II:} & \overline{U}(\tilde{\boldsymbol{L}}^2, \tilde{b}^2, \tilde{s}^2) & \longleftarrow & \overline{U}_{\nearrow}(\boldsymbol{L}^2, b^2, s^2) & \\
\vdots & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & \\
 & \boldsymbol{P} & & \boldsymbol{Q} &
\end{array}
$$

Figure 3.1: Winning strategy for the $\nearrow$-game induces one for the game

$$
\begin{array}{llccc}
 & & \text{game} & & \nearrow\text{-game} \\
\text{Player I:} & & \overline{U}(\tilde{\boldsymbol{L}}^1, \tilde{b}^1, \tilde{s}^1) & \longleftarrow & \overline{U}_{\nearrow}(\boldsymbol{L}^1, b^1, s^1) \\
 & \text{strategy} & & & \\
\text{Player II:} & & \overline{U}(\boldsymbol{K}^1, a^1, r^1) & \longrightarrow & \overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}^1, \tilde{a}^1, \tilde{r}^1) \\
\text{Player I:} & & \overline{U}(\tilde{\boldsymbol{L}}^2, \tilde{b}^2, \tilde{s}^2) & \longleftarrow & \overline{U}_{\nearrow}(\boldsymbol{L}^2, b^2, s^2) \\
 & \text{strategy} & & & \\
\text{Player II:} & & \overline{U}(\boldsymbol{K}^2, a^2, r^2) & \longrightarrow & \overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}^2, \tilde{a}^2, \tilde{r}^2) \\
\vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow \\
 & & \boldsymbol{P} & & \boldsymbol{Q}
\end{array}
$$

Figure 3.2: Winning strategy for the game induces one for the $\nearrow$-game

3.5.3 Details of the transfers

Conditions and definitions

A *stage* consists of two moves (one in the game and one in the $\nearrow$-game) which lie at the same height in Figures 3.1 and 3.2. When we describe the situation at a fixed stage, we omit the integer m indicating the stage unless ambiguity may be caused: for example, we write K_n in place of K_n^m. This is not only for simpler notation; we try to offer explanation of the transfers that will go in the proofs of both implications, and this omission solves the problem that when we describe the stage having, say K_n^m, the previous stage can have L_n^{m-1} or L_n^m depending on which implication we look at.

The transfers are executed so that the following conditions, written as $(*)$ afterwards, are fulfilled:

(1) $\tilde{a} \geq a$, $\tilde{b} \geq b$, $\tilde{r} \leq r/2$, and $\tilde{s} \leq s/2$ (in fact, $\tilde{a} = a$ and $\tilde{b} \in \{b, b+1\}$);

(2) $\bigcup_{j=1}^{n} K_j \subset \tilde{K}_n$ for $n \in [a]$, and $\bigcup_{j=1}^{n} \tilde{L}_j \subset L_n$ for $n \in [b]$;

(3) $\bigcup_{n=1}^{a} K_n = \tilde{K}_{\tilde{a}}$ and $\bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b$.

For $x \in \bigcup_{n=1}^{a} K_n = \tilde{K}_{\tilde{a}}$, its *affiliation* (n_1, n_2) is the pair of the integer $n_1 \in [a]$ with $x \in K_{n_1}$, called the *first affiliation* of x, and the least integer $n_2 \in [\tilde{a}]$ with $x \in \tilde{K}_{n_2}$, called the *second affiliation* of x. We give a similar definition for the points in $\bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b$: for $x \in \bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b$, its *affiliation* (n_1, n_2) is the pair of the integer $n_1 \in [\tilde{b}]$ with $x \in \tilde{L}_{n_1}$, called the *first affiliation* of x, and the least integer $n_2 \in [b]$ with $x \in L_{n_2}$, called the *second affiliation* of x. Strictly speaking, we should specify the stage at which the affiliations are defined, because, for instance, it may be that $L_{b^m}^m \cap L_{b^{m'}}^{m'} \neq \emptyset$ for distinct m and m'. However, since we can easily guess the stage from the context, we choose not to specify it in order to avoid complexity.

Remark 3.5.6. Condition (2) in $(*)$ is equivalent to the condition that the first affiliation is always greater than or equal to the second affiliation.

Let us look at $\overline{U}(\boldsymbol{K}, a, r) \in \mathcal{B}$ and $\overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}, \tilde{a}, \tilde{r}) \in \mathcal{B}_{\nearrow}$ at any stage except the first one. We have $\overline{U}(\tilde{\boldsymbol{L}}, \tilde{b}, \tilde{s}) \in \mathcal{B}$ and $\overline{U}_{\nearrow}(\boldsymbol{L}, b, s) \in \mathcal{B}_{\nearrow}$ at the previous stage. Since $\overline{U}(\boldsymbol{K}, a, r) \subset \overline{U}(\tilde{\boldsymbol{L}}, \tilde{b}, \tilde{s})$, for each $x \in \bigcup_{n=1}^{\tilde{b}} K_n$ there exists a unique $y \in \bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b$ satisfying $|x - y| \leq \tilde{s}$, where uniqueness follows from the assumption that any two distinct points in $\bigcup_{n=1}^{\tilde{b}} \tilde{L}_n$ have distance at least $3\tilde{s}$. This y is called the *parent* of x. Observe that if $x \in K_n$ then $y \in \tilde{L}_n$; that is to say, x and y have the same first affiliation. We give a similar definition also when we look at $\overline{U}_{\nearrow}(\boldsymbol{L}, b, s) \in \mathcal{B}_{\nearrow}$ and $\overline{U}(\tilde{\boldsymbol{L}}, \tilde{b}, \tilde{s}) \in \mathcal{B}$: the *parent* of $x \in L_{\tilde{a}}$ is the unique $y \in \bigcup_{n=1}^{a} K_n = \tilde{K}_{\tilde{a}}$ satisfying $|x - y| \leq \tilde{r}$. Observe that if $x \in L_n$ then $y \in \tilde{K}_n$; that is to say, the second affiliation of y is less than or equal to that of x. Note that x and y may have different second affiliations; for example, if $\tilde{K}_1 = \tilde{K}_2 = \{0\}$, $\tilde{a} = 2$, $\tilde{r} = 0.2$, $L_1 = \{0.01\}$, $L_2 = \{0.01, 0.1\}$, $b = 2$, and $s = 0.001$, then $x = 0.1$ has second affiliation 2, but its parent $y = 0$ has second affiliation 1.

Transfers from the game to the $\nearrow$-game

Given a move $\overline{U}(\boldsymbol{K}, a, r) \in \mathcal{B}$, we shall construct its transfer $\overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}, \tilde{a}, \tilde{r}) \in \mathcal{B}_{\nearrow}$. If it is the first move of Player I, then we put $\tilde{a} = a$, $\tilde{r} = r/2$, and $\tilde{K}_n = \bigcup_{j=1}^{n} K_j$ for every $n \in \mathbb{N}$, and we can easily see that the conditions $(*)$ are fulfilled. So suppose otherwise. Then we already know $\overline{U}_{\nearrow}(\boldsymbol{L}, b, s) \in \mathcal{B}_{\nearrow}$ and its transfer $\overline{U}(\tilde{\boldsymbol{L}}, \tilde{b}, \tilde{s}) \in \mathcal{B}$, and we have $\overline{U}(\boldsymbol{K}, a, r) \subset \overline{U}(\tilde{\boldsymbol{L}}, \tilde{b}, \tilde{s})$.

Put $\tilde{a} = a$ and $\tilde{r} = \min\{s - \tilde{s}, r/2\}$, and define $\tilde{K}_n = \bigcup_{j=1}^{n} K_j$ for $n > \tilde{b}$. For $n \leq \tilde{b}$, we let $\tilde{K}_n$ consist of those $x \in \bigcup_{n=1}^{\tilde{b}} K_n$ whose parent has the second affiliation at most n; then each $x \in \bigcup_{n=1}^{\tilde{b}} K_n$ will have the same affiliation as its parent.

Claim 1. *We have $d(\tilde{K}_n, L_n) \leq \tilde{s}$ for $n \in [b]$.*

Proof. Fix such an integer n.

Let $x \in \tilde{K}_n$ and denote its affiliation by (n_1, n_2). Then the parent y of x has affiliation (n_1, n_2) and so belongs to L_{n_2}. It follows from $y \in L_{n_2} \subset L_n$ and

$|x - y| \leq \tilde{s}$ that $x \in \overline{B}(L_n, \tilde{s})$.

Conversely, let $y \in L_n$ and denote its affiliation by (n_1, n_2). Then there exists a point $x \in K_{n_1}$ with $|x - y| \leq \tilde{s}$ because $d(K_{n_1}, \tilde{L}_{n_1}) \leq \tilde{s}$. Since y is the parent of x, the affiliation of x is (n_1, n_2). Therefore $x \in \tilde{K}_{n_2} \subset \tilde{K}_n$ and so $y \in \overline{B}(\tilde{K}_n, \tilde{s})$. ∎

We may deduce from this claim that $\overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}, \tilde{a}, \tilde{r}) \subset \overline{U}_{\nearrow}(\boldsymbol{L}, b, s)$ using the triangle inequality and $\tilde{r} + \tilde{s} \leq s$. Therefore $\overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}, \tilde{a}, \tilde{r})$ is a valid reply in the $\nearrow$-game. It is easy to see that the conditions $(*)$ are fulfilled.

Transfers from the $\nearrow$-game to the game

Given a move $\overline{U}_{\nearrow}(\boldsymbol{L}, b, s) \in \mathcal{B}_{\nearrow}$, we shall construct its transfer $\overline{U}(\tilde{\boldsymbol{L}}, \tilde{b}, \tilde{s}) \in \mathcal{B}$. If it is the first move of Player I, then we put $\tilde{b} = b$, $\tilde{s} = s/2$, and

$$\tilde{L}_n = \begin{cases} L_1 & \text{if } n = 1; \\ L_n \setminus L_{n-1} & \text{if } 2 \leq n \leq b; \\ I & \text{if } n \geq b + 1. \end{cases}$$

(Remember that the sets $\tilde{L}_n$ for $n > \tilde{b} = b$ do not have to be pairwise disjoint or finite.) We can easily see that the conditions $(*)$ are fulfilled in this case. So suppose otherwise. Then we already know $\overline{U}(\boldsymbol{K}, a, r) \in \mathcal{B}$ and its transfer $\overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}, \tilde{a}, \tilde{r}) \in \mathcal{B}_{\nearrow}$, and we have $\overline{U}_{\nearrow}(\boldsymbol{L}, b, s) \subset \overline{U}_{\nearrow}(\tilde{\boldsymbol{K}}, \tilde{a}, \tilde{r})$.

Put $\tilde{b} = b + 1$ and $\tilde{s} = \min\{r - \tilde{r}, s/2\}$, and define $\tilde{L}_n = L_{n-1}$ for $n > \tilde{b}$. We define $\tilde{L}_n$ for $n \leq \tilde{b}$ by determining the first affiliation of each point in L_b as follows. Let $x \in L_b$ and denote its second affiliation by n_2. If $n_2 > \tilde{a}$, then the first affiliation of x is n_2. Suppose $n_2 \leq \tilde{a}$, and let $y \in \tilde{K}_{n_2}$ denote the parent of x. If the second affiliation of y is n_2, then the first affiliation of x is the same as that of y; otherwise the first affiliation of x is $\tilde{b}$.

Claim 2. *We have $d(\tilde{L}_n, K_n) \leq \tilde{r}$ for $n \in [a]$.*

Proof. Fix such an integer n.

Let $x \in \tilde{L}_n$ and denote its parent by y. Then it follows that x and y have the same affiliation, and so $y \in K_n$. Hence we may infer from $|x - y| \le \tilde{r}$ that $x \in \overline{B}(K_n, \tilde{r})$.

Conversely, let $y \in K_n$ and denote its second affiliation by n_2. Then there exists a point $x \in L_{n_2}$ with $|x - y| \le \tilde{r}$ because $d(\tilde{K}_{n_2}, L_{n_2}) \le \tilde{r}$. Since y is the parent of x and has the same second affiliation as x, the first affiliation of x is n. Therefore $y \in \overline{B}(\tilde{L}_n, \tilde{r})$. ∎

We may deduce from the claim that $\overline{U}(\tilde{\boldsymbol{L}}, \tilde{b}, \tilde{s}) \subset \overline{U}(\boldsymbol{K}, a, r)$ using the triangle inequality and $\tilde{r} + \tilde{s} \le r$. Therefore $\overline{U}(\tilde{\boldsymbol{L}}, \tilde{b}, \tilde{s})$ is a valid reply in the game. It is easy to see that the conditions $(*)$ are fulfilled.

3.5.4 Proof of $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$

We shall prove that $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$, which will complete the proof of Theorem 3.2.3 due to Lemma 3.5.5. Recall that $\lim_{m\to\infty} K_n^m = P_n$ and $\lim_{m\to\infty} \tilde{K}_n^m = Q_n$ for every $n \in \mathbb{N}$.

In order to prove $\bigcup_{n=1}^{\infty} P_n \subset \bigcup_{n=1}^{\infty} Q_n$, it is enough to show that $\bigcup_{j=1}^{n} P_j \subset Q_n$ for every $n \in \mathbb{N}$. Since $\bigcup_{j=1}^{n} K_j^m \subset \tilde{K}_n^m$ for all $m \in \mathbb{N}$, we obtain $\bigcup_{j=1}^{n} P_j \subset Q_n$ by Proposition 3.1.4 (2), (3).

Now we shall prove $\bigcup_{n=1}^{\infty} Q_n \subset \bigcup_{n=1}^{\infty} P_n$. Let $x \in \bigcup_{n=1}^{\infty} Q_n$, and denote by n the least positive integer with $x \in Q_n$. Since it is easy to observe that $K_1^m = \tilde{K}_1^m$ for every $m \in \mathbb{N}$, which implies $P_1 = Q_1$, we may assume that $n \ge 2$. Because Q_{n-1} is closed and $x \notin Q_{n-1}$, there exists $r \in (0, 1)$ satisfying $\overline{B}(x, 4r) \cap Q_{n-1} = \emptyset$, that is, $x \notin \overline{B}(Q_{n-1}, 4r)$. Fix a positive integer m_0 such that $\tilde{a}^m \ge n$, $\tilde{r}^m \le r$, and $d(\tilde{K}_{n-1}^m, Q_{n-1}) \le r$ for every $m \ge m_0$. Observe that $x \notin \overline{B}(\tilde{K}_{n-1}^m, 3r)$ for every $m \ge m_0$.

Set $k_0 = \lceil 1/r \rceil$. For each $k \ge k_0$, choose $m_k \ge m_0$ satisfying $d(\tilde{K}_n^m, Q_n) \le 1/k$ for every $m \ge m_k$, and for each $m \ge m_k$ take $y_{km} \in \tilde{K}_n^m$ with $|x - y_{km}| \le 1/k$ and let $z_{km} \in \tilde{K}_n^{m_0}$ denote the unique point satisfying $|y_{km} - z_{km}| \le \tilde{r}^{m_0}$.

Claim 3. *The two points y_{km} and z_{km} have the same affiliation.*

Proof. By an *ancestor* of y_{km} we mean a point that can be written as 'the parent of … the parent of y_{km}.' Observe that z_{km} is an ancestor of y_{km}. Indeed if we denote by z'_{km} the ancestor of y_{km} in $\tilde{K}_n^{m_0}$, then

$$|y_{km} - z'_{km}| < \tilde{r}^{m_0} + \frac{\tilde{r}^{m_0}}{2} + \frac{\tilde{r}^{m_0}}{2^2} + \cdots = 2\tilde{r}^{m_0}$$

and so $|z_{km} - z'_{km}| < 3\tilde{r}^{m_0}$, which implies $z_{km} = z'_{km}$.

In order to prove our claim, it suffices to prove that the second affiliation of the ancestor $w \in \tilde{K}_n^{m'}$ of y_{km} is n for every $m' \in \{m_0, \ldots, m\}$. We can see $|w - y_{km}| \le 2\tilde{r}^{m'} \le 2r$ by the same reasoning as above. Therefore we have

$$|w - x| \le |w - y_{km}| + |y_{km} - x| \le 2r + \frac{1}{k} \le 3r.$$

It follows that the second affiliation of w cannot be less than n because $x \notin \overline{B}(\tilde{K}_{n-1}^{m'}, 3r)$. ∎

Note that all z_{km} belong to the single finite set $\tilde{K}_n^{m_0}$. We can choose $z_k \in K_n^{m_0}$ for $k \ge k_0$ inductively so that the set

$$\{\, m \ge m_k \mid z_{k_0 m} = z_{k_0}, \ldots, z_{km} = z_k \,\}$$

is infinite for any $k \ge k_0$. Then we take $z \in K_n^{m_0}$ for which $\{\, k \ge k_0 \mid z_k = z \,\}$ is infinite, and put $\{\, k \ge k_0 \mid z_k = z \,\} = \{k_1, k_2, \ldots\}$, where $k_1 < k_2 < \cdots$. Since the set

$$\{\, m \ge m_{k_j} \mid z_{k_1 m} = \cdots = z_{k_j m} = z \,\}$$

is infinite for every $j \in \mathbb{N}$, we may construct a strictly increasing sequence m'_1, m'_2, … of positive integers satisfying $z_{k_1 m'_j} = \cdots = z_{k_j m'_j} = z$.

Let l denote the first affiliation of z. Then the foregoing claim shows that whenever $i \le j$, the first affiliation of $y_{k_i m'_j}$ is l, which implies that x belongs to $\overline{B}(K_l^{m'_j}, 1/k_i)$. For any $i \in \mathbb{N}$, since $d(K_l^{m'_j}, P_l) \le 1/k_i$ for sufficiently large j, we have $x \in \overline{B}(P_l, 2/k_i)$. Hence $x \in \bigcap_{i=1}^{\infty} \overline{B}(P_l, 2/k_i) = P_l$. This completes the proof.

Chapter 4

Knot points of typical continuous functions

4.1 Statement of the main theorem

Having defined the residuality of families of F_σ sets, we are now ready to state the main theorem of this thesis. Recall that $N(f)$ denotes the set of all points in I that are not knot points of $f \in C(I)$ (see Definition 1.1.5), and that the following theorem has been announced by Zajíček [Za] and proved by Preiss and Zajíček [PZ]:

Theorem 4.1.1 ([PZ], [Za, Theorem 2.5]). *For a σ-ideal $\mathcal{I}$ on I, the following are equivalent:*

(1) *a typical function $f \in C(I)$ has the property that $N(f) \in \mathcal{I}$;*

(2) *$\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$.*

Our main theorem is the following, established by Preiss and the author:

Theorem 4.1.2 (Main Theorem, [PS]). *For a family $\mathcal{S}$ of subsets of I, the following are equivalent:*

(1) *a typical function $f \in C(I)$ has the property that $N(f) \in \mathcal{S}$.*

(2) *$\mathcal{S} \cap \mathcal{F}_\sigma$ is residual in $\mathcal{F}_\sigma$.*

Observe that Theorem 4.1.2 generalises Theorem 4.1.1 due to Proposition 3.3.4.

4.2 Basic properties of $\mathcal{K}$

Definition 4.2.1. Let $\mathscr{D}$ denote the dense subset of $\mathcal{K}^{\mathbb{N}}$ consisting of all sequences whose terms are pairwise disjoint finite sets.

Lemma 4.2.2. *If $K, L \in \mathcal{K}$ and $r > 0$ are such that $K \subset B(L, r)$, then $K \subset B(L, r - \varepsilon)$ for some $\varepsilon > 0$.*

Proof. Suppose that $K \not\subset B(L, r - \varepsilon)$ for all $\varepsilon > 0$, and take $x_n \in K \setminus B(L, r - 1/n)$ for each $n \in \mathbb{N}$. We may assume that x_n is convergent, say to x. Since $x \in K \subset B(L, r)$, there exists $y \in L$ with $|x - y| < r$. By the choice of x_n, we have $|x_n - y| \geq r - 1/n$, and so $|x - y| \geq r$, which is a contradiction. ∎

Corollary 4.2.3. *For every $r > 0$, the set $\{(K, L) \in \mathcal{K}^2 \mid K \subset B(L, r)\}$ is open in $\mathcal{K}^2$.*

Proof. Let (K_0, L_0) belong to the set, and take $\varepsilon > 0$ with $K_0 \subset B(L_0, r - \varepsilon)$ using the previous lemma. If $(K, L) \in \mathcal{K}^2$ satisfies $d(K, K_0) < \varepsilon/2$ and $d(L, L_0) < \varepsilon/2$, then

$$K \subset B(K_0, \varepsilon/2) \subset B(L_0, r - \varepsilon/2) \subset B(L, r).$$

This completes the proof. ∎

4.3 Basic properties of $N(f,a)$

4.3.1 Definition of $N(f,a)$

Definition 4.3.1. For $f \in C(I)$ and $a > 0$, we define

$$N^+(f,a) = \{x \in [0, 1-2^{-a}] \mid f(y) - f(x) \le a(y-x) \text{ for all } y \in [x, x+2^{-a}]\},$$
$$N_+(f,a) = \{x \in [0, 1-2^{-a}] \mid f(y) - f(x) \ge -a(y-x) \text{ for all } y \in [x, x+2^{-a}]\},$$
$$N^-(f,a) = \{x \in [2^{-a}, 1] \mid f(y) - f(x) \ge a(y-x) \text{ for all } y \in [x-2^{-a}, x]\},$$
$$N_-(f,a) = \{x \in [2^{-a}, 1] \mid f(y) - f(x) \le -a(y-x) \text{ for all } y \in [x-2^{-a}, x]\},$$

and

$$\begin{aligned}\hat{N}(f,a) &= N^+(f,a) \cup N_-(f,a),\\ \check{N}(f,a) &= N_+(f,a) \cup N^-(f,a),\\ N(f,a) &= \hat{N}(f,a) \cup \check{N}(f,a)\\ &= N^+(f,a) \cup N_+(f,a) \cup N^-(f,a) \cup N_-(f,a).\end{aligned}$$

Convention 4.3.2. We shall use the symbol $\tilde{N}$ in a statement to mean that the statement with $\tilde{N}$ replaced by $\hat{N}$ and the statement with $\tilde{N}$ replaced by $\check{N}$ are both true; for instance, by $\tilde{N}(f,a) \subset \tilde{N}(g,b)$ we mean $\hat{N}(f,a) \subset \hat{N}(g,b)$ and $\check{N}(f,b) \subset \check{N}(g,b)$.

Remark 4.3.3. The mean value theorem shows that

$$|2^{-a} - 2^{-b}| \le |a-b| \log 2 \le |a-b|$$

for all $a, b > 0$. This estimate will sometimes be used implicitly in this thesis.

Proposition 4.3.4. *If $f \in C(I)$ and $0 < a_1 < a_2 < \cdots \to \infty$, then $N(f) = \bigcup_{n=1}^{\infty} N(f, a_n)$.*

Proof. Trivial. ∎

4.3.2 Descriptive properties of knot points

Proposition 4.3.5. *For every $f \in C(I)$ and $a > 0$, the sets $N^{\pm}(f,a)$, $N_{\pm}(f,a)$, $\tilde{N}(f,a)$, and $N(f,a)$ are all closed. Therefore $N(f)$ is F_σ for every $f \in C(I)$.*

Proof. Obviously it suffices to show that $N^+(f,a)$ is closed. Suppose that a sequence x_n of points in $N^+(f,a)$ converges to a point x. Since $x_n \in [0, 1-2^{-a}]$ for all $n \in \mathbb{N}$, we have $x \in [0, 1-2^{-a}]$. Assume for a contradiction that $f(y) - f(x) > a(y-x)$ for some $y \in [x, x+2^{-a}]$. By the continuity of f, we may assume that $y \in (x, x+2^{-a})$. Then since x_n converges to x and f is continuous, there exists $n \in \mathbb{N}$ such that $y \in (x_n, x_n + 2^{-a})$ and $f(y) - f(x_n) > a(y - x_n)$, which contradicts x_n belonging to $N^+(f,a)$. ∎

It is natural to ask whether $N(f)$ being F_σ is the best possible result. Obviously $N(f)$ does belong to a lower descriptive class for some $f \in C(I)$; for example, $N(f) = I$ if f is differentiable. However, it turns out that $N(f)$ is not G_δ for typical functions. We include its easy proof for completeness, though we do not use this result afterwards.

Proposition 4.3.6. *For a typical function $f \in C(I)$, the set $N(f)$ is not G_δ, i.e. $N(f)$ is true F_σ.*

Proof. By Theorem 1.1.4, it is enough to prove that $N(f)$ is dense in I for every $f \in C(I)$ and that no dense null F_σ subset of I is G_δ.

The former follows from the observation that for each nondegenerate closed subinterval J of I, any point at which f restricted to J attains its maximum must belong to $N(f)$.

To show the latter, consider any dense null F_σ subset F of I, and write $F = \bigcup_{n=1}^{\infty} F_n$ with closed sets F_n. For every n, since F_n is null, it must have empty interior, and so it must be nowhere dense. Hence F is meagre. If F were G_δ, then F would be residual because F is dense. Therefore F is not G_δ. ∎

By Proposition 4.3.5, we can restate our main theorem (Theorem 4.1.2) as follows:

Theorem 4.3.7 (Main Theorem). *For a subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$, the following are equivalent*:

(1) *a typical function $f \in C(I)$ has the property that $N(f) \in \mathcal{F}$;*

(2) *$\mathcal{F}$ is residual.*

4.3.3 Continuity of $N(f,a)$

Proposition 4.3.8. *Suppose that $0 < a < b$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that whenever $f, g \in C(I)$ satisfy $\|f - g\| < \delta$, we have $\tilde{N}(f,a) \subset B\big(\tilde{N}(g,b),\varepsilon\big)$ and $N(f,a) \subset B\big(N(g,b),\varepsilon\big)$.*

Proof. We may assume that $\varepsilon < 2^{-a} - 2^{-b}$ without loss of generality. Choose $\delta > 0$ with $\delta < \varepsilon(b-a)/2$. We shall show that this δ satisfies the required condition. It suffices to prove that $N^+(f,a) \subset B\big(N^+(g,b),\varepsilon\big)$.

Take any $x \in N^+(f,a)$, and let $y_0 \in [x, x+2^{-a}]$ be a point at which the continuous function $y \longmapsto g(y) - by$ defined on $[x, x+2^{-a}]$ attains its maximum. It is enough to show that $x \le y_0 < x + \varepsilon$ and $y_0 \in N^+(g,b)$.

The definition of y_0 gives $g(y_0) - by_0 \ge g(x) - bx$, which implies

$$b(y_0 - x) \le g(y_0) - g(x) < f(y_0) - f(x) + 2\delta \le a(y_0 - x) + 2\delta$$

because $x \in N^+(f,a)$ and $y_0 \in [x, x+2^{-a}]$. It follows that $y_0 - x < 2\delta/(b-a) < \varepsilon$.

With the aim of proving $y_0 \in N^+(g,b)$, take any $y \in [y_0, y_0 + 2^{-b}]$. Since

$$x \le y_0 \le y \le y_0 + 2^{-b} < x + \varepsilon + 2^{-b} < x + 2^{-a},$$

the definition of y_0 again gives $g(y_0) - by_0 \ge g(y) - by$, or equivalently $g(y) - g(y_0) \le b(y - y_0)$. This completes the proof. ∎

4.3.4 Properties of continuously differentiable functions

Lemma 4.3.9. *If $f \in C^1(I)$ and $0 < a < b$, then there exists $\delta > 0$ such that $B\big(\tilde{N}(f,a),\delta\big) \subset \tilde{N}(f,b)$.*

Proof. By symmetry, it suffices to show that $B\big(N^+(f,a),\delta\big) \subset N^+(f,b)$ for some $\delta > 0$. Suppose that this is false. For each $n \in \mathbb{N}$, let $\delta_n = (2^{-a} - 2^{-b})/n$ and take $x_n \in B\big(N^+(f,a),\delta_n\big) \setminus N^+(f,b)$. We may assume that x_n converges, say to x. Observe that

$$x \in \bigcap_{n=1}^{\infty} B\big(N^+(f,a), \delta_n + |x - x_n|\big) = N^+(f,a).$$

Since

$$\begin{aligned} x_n &\in B\big(N^+(f,a),\delta_n\big) \setminus N^+(f,b) \\ &\subset B([0,1-2^{-a}], 2^{-a} - 2^{-b}) \setminus N^+(f,b) \\ &\subset [0,1-2^{-b}] \setminus N^+(f,b), \end{aligned}$$

we may take $y_n \in (x_n, x_n + 2^{-b}]$ with $f(y_n) - f(x_n) > b(y_n - x_n)$. We may assume that y_n converges, say to y. The continuity of f shows that $f(y) - f(x) \geq b(y-x)$, whereas we have $f(y) - f(x) \leq a(y-x)$ because $x \in N^+(f,a)$ and $x \leq y \leq x + 2^{-b} < x + 2^{-a}$. It follows that $y = x$.

By the mean value theorem, we may take $z_n \in (x_n, y_n)$ with

$$f'(z_n) = \frac{f(y_n) - f(x_n)}{y_n - x_n} > b.$$

Since both x_n and y_n converge to x, so does z_n. The continuity of f' shows that $f'(x) \geq b$, which contradicts $x \in N^+(f,a)$. ∎

Corollary 4.3.10. *If $f \in C^1(I)$ and $0 < a < b$, then $\tilde{N}(f,a) \subset \operatorname{Int} \tilde{N}(f,b)$.*

Proof. Immediate from Lemma 4.3.9. ∎

Proposition 4.3.11. *Suppose that $f \in C^1(I)$ and $0 < a < b$. Then there exists $\delta > 0$ such that $B\big(\tilde{N}(g,a),\delta\big) \subset \tilde{N}(f,b)$ for every $g \in B(f,\delta)$.*

Proof. Set $c = (a+b)/2$, so that $0 < a < c < b$. By Lemma 4.3.9 we may find $\varepsilon > 0$ with $B\big(\tilde{N}(f,c),2\varepsilon\big) \subset \tilde{N}(f,b)$, and by Proposition 4.3.8 we may find $\tau > 0$ such that $\tilde{N}(g,a) \subset B\big(\tilde{N}(f,c),\varepsilon\big)$ for all $g \in B(f,\tau)$. We set $\delta = \min\{\varepsilon,\tau\}$. Then for every $g \in B(f,\delta)$, we have

$$B\big(\tilde{N}(g,a),\delta\big) \subset B\big(\tilde{N}(f,c),\delta+\varepsilon\big) \subset B\big(\tilde{N}(f,c),2\varepsilon\big) \subset \tilde{N}(f,b).$$ ∎

Lemma 4.3.12. *Suppose that $f \in C^1(I)$ and $0 < a < c < b$. Then there exists $\varepsilon > 0$ such that for each $x \in [0, 1-2^{-a}] \setminus N^+(f,b)$, we may find $y \in (x+\varepsilon, x+2^{-b}]$ with $f(y) - f(x) > c(y-x)$.*

Proof. Suppose that the lemma is false. Then for each $n \in \mathbb{N}$, we may find $x_n \in [0, 1-2^{-a}] \setminus N^+(f,b)$ such that $f(y) - f(x_n) \leq c(y - x_n)$ for all $y \in (x_n + 1/n, x_n + 2^{-b}]$. We may assume that x_n converges, say to $x \in [0, 1-2^{-a}] \subset [0, 1-2^{-b}]$.

Firstly, we prove that $f(y) - f(x) \leq c(y-x)$ for all $y \in (x, x+2^{-b})$. Fix such y. For sufficiently large n, since $y \in (x_n + 1/n, x_n + 2^{-b})$, we have $f(y) - f(x_n) \leq c(y - x_n)$ by the choice of x_n. Letting $n \to \infty$, we obtain $f(y) - f(x) \leq c(y-x)$.

Now, it follows that $f'(x) \leq c$, and so $f' \leq b$ in some neighbourhood of x because $f \in C^1(I)$. Take $n \in \mathbb{N}$ so large that the interval $[x_n, x_n + 1/n]$ is contained in the neighbourhood. Then the mean value theorem shows that $f(y) - f(x_n) \leq b(y - x_n)$ for all $y \in [x_n, x_n + 1/n]$. This, together with the choice of x_n, implies that $x_n \in N^+(f,b)$, a contradiction. ∎

Proposition 4.3.13. *Suppose that $f \in C^1(I)$ and $0 < a < b$. Then there exists $l > 0$ such that every set of one of the following forms contains an open interval of length l:*

(1) $\{y \in [x, x+2^{-a}] \mid f(y) - f(x) > a(y-x)\}$ *for* $x \in [0, 1-2^{-a}] \setminus N^+(f,b)$;

(2) $\{y \in [x, x+2^{-a}] \mid f(y) - f(x) < -a(y-x)\}$ *for* $x \in [0, 1-2^{-a}] \setminus N_+(f,b)$;

(3) $\{y \in [x-2^{-a}, x] \mid f(y) - f(x) < a(y-x)\}$ *for* $x \in [2^{-a}, 1] \setminus N^-(f,b)$;

(4) $\{y \in [x-2^{-a}, x] \mid f(y) - f(x) > -a(y-x)\}$ *for* $x \in [2^{-a}, 1] \setminus N_-(f,b)$.

Proof. Set $c = (a+b)/2$ and choose $\varepsilon > 0$ as in Lemma 4.3.12. Then take $l > 0$ so that $l/2 < \min\{\varepsilon, 2^{-a} - 2^{-b}\}$ and $(\|f'\| + a)l/2 < (c-a)\varepsilon$. We shall show that this l satisfies the required condition. By symmetry, we only need to look at sets of the first form.

Let $x \in [0, 1-2^{-a}] \setminus N^+(f,b)$ and set

$$S = \{y \in [x, x+2^{-a}] \mid f(y) - f(x) > a(y-x)\}.$$

By the choice of ε, we may find $t \in (x+\varepsilon, x+2^{-b}]$ with $f(t) - f(x) > c(t-x)$. It suffices to show that S contains the open interval $(t-l/2, t+l/2)$. If $y \in (t-l/2, t+l/2)$, then since

$$\begin{aligned} y &> t - l/2 > x + \varepsilon - l/2 > x, \\ y &< t + l/2 \leq x + 2^{-b} + l/2 < x + 2^{-a}, \end{aligned}$$

and

$$\begin{aligned} f(y) - f(x) - a(y-x) &= \big(f(y) - f(t)\big) + \big(f(t) - f(x)\big) - a(y-x) \\ &> -\|f'\||y-t| + c(t-x) - a(y-x) \\ &= (c-a)(t-x) - \|f'\||y-t| - a(y-t) \\ &\geq (c-a)(t-x) - (\|f'\| + a)|y-t| \\ &> (c-a)\varepsilon - (\|f'\| + a)l/2 \\ &> 0, \end{aligned}$$

it follows that $y \in S$. ∎

4.3.5 Bump functions

Definition 4.3.14. Let $\hat{H}$ and $\check{H}$ be disjoint finite subsets of I, and h and w be positive numbers. A *bump function* of height h and width w located at $\hat{H}$ and $\check{H}$ is a function $\varphi \in C^1(I)$ with the following properties:

- $\|\varphi\| = h$;
- $\varphi(x) = h$ for all $x \in \hat{H}$ and $\varphi(x) = -h$ for all $x \in \check{H}$;
- $\{x \in I \mid \varphi(x) > 0\} \subset B(\hat{H}, w)$ and $\{x \in I \mid \varphi(x) < 0\} \subset B(\check{H}, w)$.

Remark 4.3.15. If $\hat{H}$, $\check{H}$, h, and w satisfy the conditions at the beginning of the definition above, there exists a bump function of height h and width w located at $\hat{H}$ and $\check{H}$.

Proposition 4.3.16. *Let $f \in C(I)$ and $a > 0$. Suppose that φ is a bump function of height $h > 0$ and width $w > 0$ located at $\hat{H}$ and $\check{H}$, where $\hat{H}$ and $\check{H}$ are disjoint finite subsets of I. Then, setting $g = f + \varphi$, we have $\tilde{H} \cap \tilde{N}(f, a) \subset \tilde{N}(g, a)$.*

Proof. It suffices to show that $\hat{H} \cap \hat{N}(f, a) \subset \hat{N}(g, a)$. Let $x \in \hat{H} \cap \hat{N}(f, a)$. Then $x \in N^+(f, a) \cup N_-(f, a)$, and we may assume that $x \in N^+(f, a)$ by symmetry. We have $x \in [0, 1 - 2^{-a}]$ by the definition of $N^+(f, a)$; if $y \in [x, x + 2^{-a}]$, then

$$g(y) - g(x) = \big(f(y) + \varphi(y)\big) - \big(f(x) + h\big) \leq f(y) - f(x) \leq a(y - x).$$

It follows that $x \in N^+(g, a)$. ∎

Proposition 4.3.17. *Suppose that $f \in C^1(I)$, $0 < a < b$, and $h > 0$. Then there exists $\mu > 0$ with the following property:*

> *Suppose that φ is a bump function of height h and width $w > 0$ located at $\hat{H}$ and $\check{H}$, where $\hat{H}$ and $\check{H}$ are disjoint finite subsets of I satisfying $B(\tilde{H}, \mu) = I$. Then, setting $g = f + \varphi$, we have $\tilde{N}(g, a) \subset \tilde{N}(f, b) \cap B(\tilde{H}, w)$.*

Proof. Choose $l > 0$ as in Proposition 4.3.13. Take $\mu > 0$ so small that $\mu < l/2$, $2\mu < 2^{-a}$, and $2\mu(\|f'\| + a) < h$. We shall show that this μ satisfies the required condition. Let φ and g be as in the statement. By symmetry, it suffices to show that $N^+(g, a) \subset N^+(f, b) \cap B(\hat{H}, w)$. Let $x \in N^+(g, a)$.

Firstly, we show that $x \in N^+(f, b)$. Assume, to derive a contradiction, that $x \notin N^+(f, b)$. Then, since

$$x \in N^+(g, a) \setminus N^+(f, b) \subset [0, 1 - 2^{-a}] \setminus N^+(f, b),$$

the set $\{y \in [x, x + 2^{-a}] \mid f(y) - f(x) > a(y - x)\}$ contains an open interval of length l. Because $B(\hat{H}, l/2) \supset B(\hat{H}, \mu) = I$, we may find $y \in \hat{H}$ such that $y \in [x, x + 2^{-a}]$ and $f(y) - f(x) > a(y - x)$. Then

$$g(y) - g(x) = \big(f(y) + h\big) - \big(f(x) + \varphi(x)\big) \geq f(y) - f(x) > a(y - x),$$

which contradicts the assumption that $x \in N^+(g, a)$.

Secondly, we show that $x \in B(\hat{H}, w)$. Because $B(\hat{H}, \mu) = I$, we may find $y \in [x, x + 2\mu] \cap \hat{H}$. Then

$$\begin{aligned} a(y - x) &\geq g(y) - g(x) = \big(f(y) + h\big) - \big(f(x) + \varphi(x)\big) \\ &\geq h - \varphi(x) - \|f'\|(y - x), \end{aligned}$$

which implies that

$$\varphi(x) \geq h - (\|f'\| + a)(y - x) \geq h - 2\mu(\|f'\| + a) > 0.$$

It follows that $x \in B(\hat{H}, w)$. ∎

Definition 4.3.18. If $f \in C^1(I)$, $0 < a < b$ and $h > 0$, then $\mu(f, a, b, h)$ denotes a positive number μ with the property in Proposition 4.3.17.

4.4 A topological zero-one law and a key proposition

4.4.1 A topological zero-one law

Definition 4.4.1. Let X be a set. A subset A of $X^{\mathbb{N}}$ is said to be *invariant under finite permutations* if for every permutation σ on $\mathbb{N}$ that fixes all but finitely many positive integers and for every $\boldsymbol{x} \in A$, we have $(x_{\sigma(n)}) \in A$.

Proposition 4.4.2 ([Ke, Theorem 8.46]). *Let X be a Baire space and G a group of homeomorphisms on X with the property that for every pair of nonempty open subsets U and V of X, there exists $\varphi \in G$ such that $\varphi(U) \cap V \neq \emptyset$. Suppose that a subset A of X has the Baire property and that $\varphi(A) = A$ for every $\varphi \in G$. Then A is either meagre or residual.*

Remark 4.4.3. If G is a group of bijections on a set X and A is a subset of X, then the condition that $\varphi(A) = A$ for all $\varphi \in G$ is equivalent to the condition that $\varphi(A) \subset A$ for all $\varphi \in G$.

Proposition 4.4.4. *Let X be a Baire space and A a subset of $X^{\mathbb{N}}$ that is invariant under finite permutations and has the Baire property. Then A is either meagre or residual.*

Proof. Since the proposition is obvious if $X = \emptyset$, we may assume that $X \neq \emptyset$ and take an element $a \in X$.

For each permutation σ on $\mathbb{N}$, let φ_σ be the homeomorphism on $X^{\mathbb{N}}$ defined by $\varphi_\sigma(\boldsymbol{x}) = (x_{\sigma(n)})$ for $\boldsymbol{x} \in X^{\mathbb{N}}$. Write G for the set of all φ_σ where σ is a permutation that fixes all but finitely many positive integers. It is obvious that G is a group. In the light of Proposition 4.4.2, it suffices to show that for every pair of nonempty open subsets U and V of $X^{\mathbb{N}}$, there exists $\varphi \in G$ such that $\varphi(U) \cap V \neq \emptyset$.

Let U and V be nonempty open subsets of $X^{\mathbb{N}}$. Take $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$, and choose $m \in \mathbb{N}$ so that $\boldsymbol{x} \in U$ if $x_n = u_n$ for all $n \in [m]$, and $\boldsymbol{x} \in V$ if $x_n = v_n$ for all $n \in [m]$. Define a permutation σ on $\mathbb{N}$ by setting

$$\sigma(n) = \begin{cases} n+m & \text{for } n \in [m]; \\ n-m & \text{for } n \in [2m] \setminus [m]; \\ n & \text{for } n \in \mathbb{N} \setminus [2m]. \end{cases}$$

Then σ fixes all integers greater than $2m$, and so $\varphi_\sigma \in G$. Moreover, φ_σ satisfies $\varphi_\sigma(U) \cap V \neq \emptyset$ because $(u_1, \dots, u_m, v_1, \dots, v_m, a, a, \dots) \in U$ and

$$\varphi_\sigma\big((u_1, \dots, u_m, v_1, \dots, v_m, a, a, \dots)\big) = (v_1, \dots, v_m, u_1, \dots, u_m, a, a, \dots) \in V.$$

This completes the proof. ∎

4.4.2 Definition and basic properties of $\mathscr{X}$

Definition 4.4.5. (1) We put

$$X = \{\boldsymbol{a} \in (0,\infty)^{\mathbb{N}} \mid a_1 < a_2 < \cdots \to \infty\},$$

$$Y = \{\boldsymbol{\delta} \in (0,1)^{\mathbb{N}} \mid \delta_1 > \delta_2 > \cdots \to 0\},$$

$$Z = \{\boldsymbol{n} \in \mathbb{N}^{\mathbb{N}} \mid n_{j+1} \geq n_j + j \text{ for all } j \in \mathbb{N}\}.$$

These are Polish spaces in the relative topology because they are G_δ subsets of the Polish spaces $(0,\infty)^{\mathbb{N}}$, $(0,1)^{\mathbb{N}}$, and $\mathbb{N}^{\mathbb{N}}$ respectively (see Proposition 2.3.3 (2)).

(2) For $\boldsymbol{n} \in Z$ and $j, m \in \mathbb{N}$ with $j \leq m$, we define a finite subset $A_j^m(\boldsymbol{n})$ of $\mathbb{N}$ by

$$A_j^m(\boldsymbol{n}) = [n_j] \cup \bigcup_{i=j}^{m-1} \{n_i + 1, \dots, n_i + j - 1\}.$$

For $\boldsymbol{n} \in Z$ and $k \in \mathbb{Z}_+$, we define $\boldsymbol{n}^k \in Z$ by setting $n_j^k = n_{j+k}$ for $j \in \mathbb{N}$.

(3) Let $\boldsymbol{n} \in Z$ and $\boldsymbol{\delta} \in Y$. For $k \in \mathbb{Z}_+$, we define $\mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta})$ as the set of all $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}$ such that

$$\bigcup_{n \in A_j^m(\boldsymbol{n}^k) \setminus A_j^{m-1}(\boldsymbol{n}^k)} K_n \subset \bigcup_{n \in A_{j-1}^{m-1}(\boldsymbol{n}^k)} B(K_n, \delta_m)$$

whenever $2 \leq j \leq m-1$. In addition we define $\mathscr{S}(\boldsymbol{n}, \boldsymbol{\delta}) = \bigcup_{k=0}^{\infty} \mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta})$.

Remark 4.4.6. To be precise, the definition of $A_j^m(\boldsymbol{n})$ is as follows:

$$A_j^m(\boldsymbol{n}) = \begin{cases} [n_j] & \text{if } j = 1 \text{ or } j = m; \\ [n_j] \cup \bigcup_{i=j}^{m-1} \{n_i + 1, \dots, n_i + j - 1\} & \text{if } 2 \leq j \leq m-1. \end{cases}$$

Remark 4.4.7. For the reader's convenience, we spell out $A_j^m(\boldsymbol{n})$ for small j and m, writing $A_j^m = A_j^m(\boldsymbol{n})$ for simplicity:

(1) if $j = 1$, then $A_1^m = [n_1]$ for all $m \in \mathbb{N}$;

(2) if $j = 2$, then $A_2^2 = [n_2]$, $A_2^3 = [n_2 + 1]$, $A_2^4 = [n_2 + 1] \cup \{n_3 + 1\}$, $A_2^5 = [n_2 + 1] \cup \{n_3 + 1, n_4 + 1\}$ and so forth;

(3) if $j = 3$, then $A_3^3 = [n_3]$, $A_3^4 = [n_3 + 2]$, $A_3^5 = [n_3 + 2] \cup \{n_4 + 1, n_4 + 2\}$, $A_3^6 = [n_3 + 2] \cup \{n_4 + 1, n_4 + 2, n_5 + 1, n_5 + 2\}$ and so forth.

Remark 4.4.8. Note that $A_j^m(\boldsymbol{n})$ depends only on n_k for $k \in [\max\{j, m-1\}]$; in particular, $A_j^m(\boldsymbol{n}) = A_j^m(\boldsymbol{n}')$ if $n_k = n_k'$ for all $k \in [m]$.

Proposition 4.4.9. *Let $\boldsymbol{n} \in Z$, $\boldsymbol{\delta} \in Y$, and $k \in \mathbb{Z}_+$.*

(1) $[n_j] = A_j^j(\boldsymbol{n}) \subset A_j^{j+1}(\boldsymbol{n}) \subset A_j^{j+2}(\boldsymbol{n}) \subset \cdots$ *for every* $j \in \mathbb{N}$, *and* $[n_1] = A_1^m(\boldsymbol{n}) \subset \cdots \subset A_m^m(\boldsymbol{n}) = [n_m]$ *for every* $m \in \mathbb{N}$. *In particular,* $[n_j] \subset A_j^m(\boldsymbol{n}) \subset [n_m]$ *whenever* $j \leq m$.

(2) $A_j^{m+1}(\boldsymbol{n}^k) \subset A_j^m(\boldsymbol{n}^{k+1})$ *for all* $j, m \in \mathbb{N}$ *with* $j \leq m$.

(3) $\mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta}) = \mathscr{S}_0(\boldsymbol{n}^k, \boldsymbol{\delta})$.

(4) $\mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta}) \subset \mathscr{S}_{k+1}(\boldsymbol{n}, \boldsymbol{\delta})$.

Proof. (1) Immediate from the definition.

(2) We have

$$\begin{aligned}
A_j^m(\boldsymbol{n}^{k+1}) &= [n_j^{k+1}] \cup \bigcup_{i=j}^{m-1} \{n_i^{k+1} + 1, \ldots, n_i^{k+1} + j - 1\} \\
&= [n_{j+1}^k] \cup \bigcup_{i=j}^{m-1} \{n_{i+1}^k + 1, \ldots, n_{i+1}^k + j - 1\} \\
&\supset [n_j^k + j - 1] \cup \bigcup_{i=j+1}^{m} \{n_i^k + 1, \ldots, n_i^k + j - 1\} \\
&= [n_j^k] \cup \bigcup_{i=j}^{m} \{n_i^k + 1, \ldots, n_i^k + j - 1\} \\
&= A_j^{m+1}(\boldsymbol{n}^k).
\end{aligned}$$

(3) Immediate from the definition.

(4) Suppose that $\boldsymbol{K} \in \mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta})$ and $2 \leq j \leq m-1$. Then we have

$$\begin{aligned}
A_j^m(\boldsymbol{n}^{k+1}) \setminus A_j^{m-1}(\boldsymbol{n}^{k+1}) &= \{n_{m-1}^{k+1} + 1, \ldots, n_{m-1}^{k+1} + j - 1\} \\
&= \{n_m^k + 1, \ldots, n_m^k + j - 1\} \\
&= A_j^{m+1}(\boldsymbol{n}^k) \setminus A_j^m(\boldsymbol{n}^k),
\end{aligned}$$

which, together with (2), implies that

$$\begin{aligned}\bigcup_{n\in A_j^m(\boldsymbol{n}^{k+1})\setminus A_j^{m-1}(\boldsymbol{n}^{k+1})} K_n &= \bigcup_{n\in A_j^{m+1}(\boldsymbol{n}^{k})\setminus A_j^{m}(\boldsymbol{n}^{k})} K_n \\ &\subset \bigcup_{n\in A_{j-1}^m(\boldsymbol{n}^k)} B(K_n,\delta_{m+1}) \\ &\subset \bigcup_{n\in A_{j-1}^{m-1}(\boldsymbol{n}^{k+1})} B(K_n,\delta_m)\end{aligned}$$

because $\delta_{m+1} < \delta_m$. Hence we obtain $\boldsymbol{K} \in \mathscr{S}_{k+1}(\boldsymbol{n},\boldsymbol{\delta})$. ∎

Proposition 4.4.10. *Let $\boldsymbol{n} \in Z$, $\boldsymbol{\delta} \in Y$, and $k \in \mathbb{Z}_+$. If $\boldsymbol{K} \in \mathscr{S}_k(\boldsymbol{n},\boldsymbol{\delta})$, then*

$$\bigcap_{m=j}^{\infty} \bigcup_{n\in A_j^m(\boldsymbol{n}^k)} B(K_n,\delta_m) \subset \bigcup_{n=1}^{\infty} K_n$$

for all $j \in \mathbb{N}$.

Proof. By Proposition 4.4.9 (3), we may assume that $k = 0$. For simplicity we write A_j^m for $A_j^m(\boldsymbol{n})$. Fix $j \in \mathbb{N}$ and take any $x \in \bigcap_{m=j}^{\infty}\bigcup_{n\in A_j^m} B(K_n,\delta_m)$. Seeking a contradiction, suppose that $x \notin \bigcup_{n=1}^{\infty} K_n$.

For each $i \in \mathbb{N}$, set $A_i = \bigcup_{m=i}^{\infty} A_i^m$ and $L_i = \overline{\bigcup_{n\in A_i} K_n}$. Then we have

$$x \in \bigcap_{m=j}^{\infty} \bigcup_{n\in A_j^m} B(K_n,\delta_m) \subset \bigcap_{m=j}^{\infty} B(L_j,\delta_m) = L_j,$$

which allows us to define i_0 as the minimum $i \in \mathbb{N}$ with $x \in L_i$.

If $i_0 = 1$, then $A_1 = [n_1]$ and $x \in L_1 = \bigcup_{n=1}^{n_1} K_n$, contradicting our assumption that $x \notin \bigcup_{n=1}^{\infty} K_n$. Thus $i_0 \geq 2$.

For each $m \in \mathbb{N}$, take $x_m \in \bigcup_{n\in A_{i_0}} K_n$ with $|x_m - x| < 1/m$ and choose $k_m \in A_{i_0}$ with $x_m \in K_{k_m}$. If there exists $k \in \mathbb{N}$ such that $k_m = k$ for infinitely many $m \in \mathbb{N}$, then $x = \lim_{m\to\infty} x_m \in K_k$, contradicting our assumption; therefore such k does not exist. Consequently, for each $i \geq i_0$, we may take $m_i \in \mathbb{N}$ with $k_{m_i} \notin A_{i_0}^i$, and we may assume that $m_{i_0} < m_{i_0+1} < \cdots \to \infty$. Then for each

$i \geq i_0$ we have

$$\begin{aligned} x_{m_i} \in K_{k_{m_i}} &\subset \bigcup_{n \in A_{i_0} \setminus A_{i_0}^i} K_n = \bigcup_{l=i+1}^{\infty} \bigcup_{n \in A_{i_0}^l \setminus A_{i_0}^{l-1}} K_n \\ &\subset \bigcup_{l=i+1}^{\infty} \bigcup_{n \in A_{i_0-1}^{l-1}} B(K_n, \delta_l) \subset \bigcup_{l=i+1}^{\infty} \bigcup_{n \in A_{i_0-1}^{l-1}} B(K_n, \delta_{i+1}) \\ &\subset \bigcup_{n \in A_{i_0-1}} B(K_n, \delta_{i+1}) \subset B(L_{i_0-1}, \delta_{i+1}), \end{aligned}$$

keeping in mind that $\boldsymbol{K} \in \mathscr{S}_0(\boldsymbol{n}, \boldsymbol{\delta})$ and $\boldsymbol{\delta} \in Y$. It follows that

$$x \in \bigcap_{i=i_0}^{\infty} B(L_{i_0-1}, \delta_{i+1} + 1/m_i) = L_{i_0-1},$$

which violates the minimality of i_0. This completes the proof. ∎

Definition 4.4.11. For $k \in \mathbb{Z}_+$, we define $\mathscr{Y}_k$ as the set of all

$$(\boldsymbol{K}, f, \boldsymbol{n}, \boldsymbol{\delta}, \boldsymbol{a}, \boldsymbol{b}) \in \mathcal{K}^{\mathbb{N}} \times C(I) \times Z \times Y \times X \times X$$

such that $\boldsymbol{K} \in \mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta})$ and

$$N(f, a_j) \subset \bigcup_{n \in A_j^m(\boldsymbol{n}^k)} B(K_n, \delta_m), \qquad \bigcup_{n \in A_j^m(\boldsymbol{n})} K_n \subset B\big(N(f, b_j), \delta_m\big)$$

whenever $j \leq m$. Set $\mathscr{Y} = \bigcup_{k=0}^{\infty} \mathscr{Y}_k$ and write $\mathscr{X}$ for the projection of $\mathscr{Y}$ to $\mathcal{K}^{\mathbb{N}} \times C(I)$.

Remark 4.4.12. Note the difference between the subscripts of the two unions above.

Proposition 4.4.13. *We have $\mathscr{Y}_k \subset \mathscr{Y}_{k+1}$ for all $k \in \mathbb{Z}_+$.*

Proof. Thanks to Proposition 4.4.9 (4), it suffices to prove that

$$\bigcup_{n \in A_j^{m+1}(\boldsymbol{n}^k)} B(K_n, \delta_{m+1}) \subset \bigcup_{n \in A_j^m(\boldsymbol{n}^{k+1})} B(K_n, \delta_m)$$

whenever $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}$, $\boldsymbol{n} \in Z$, $\boldsymbol{\delta} \in Y$, and $j \leq m$. Proposition 4.4.9 (2) shows that

$$\begin{aligned} \bigcup_{n \in A_j^m(\boldsymbol{n}^{k+1})} B(K_n, \delta_m) &\supset \bigcup_{n \in A_j^{m+1}(\boldsymbol{n}^k)} B(K_n, \delta_m) \\ &\supset \bigcup_{n \in A_j^{m+1}(\boldsymbol{n}^k)} B(K_n, \delta_{m+1}) \end{aligned}$$

because $\delta_m > \delta_{m+1}$. ∎

Proposition 4.4.14. *If* $(\boldsymbol{K}, f) \in \mathscr{X}$, *then* $\bigcup_{n=1}^{\infty} K_n = N(f)$.

Proof. Take $\boldsymbol{n} \in Z$, $\boldsymbol{\delta} \in Y$, $\boldsymbol{a}, \boldsymbol{b} \in X$, and $k \in \mathbb{Z}_+$ so that $(\boldsymbol{K}, f, \boldsymbol{n}, \boldsymbol{\delta}, \boldsymbol{a}, \boldsymbol{b}) \in \mathscr{Y}_k$.

Firstly, we prove that $\bigcup_{n=1}^{\infty} K_n \subset N(f)$. Since

$$\bigcup_{n=1}^{n_j} K_n = \bigcap_{m=j}^{\infty} \bigcup_{n \in A_j^m(\boldsymbol{n})} K_n \subset \bigcap_{m=j}^{\infty} B\big(N(f, b_j), \delta_m\big) = N(f, b_j)$$

for every $j \in \mathbb{N}$, we have

$$\bigcup_{n=1}^{\infty} K_n = \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{n_j} K_n \subset \bigcup_{j=1}^{\infty} N(f, b_j) = N(f).$$

Secondly, we prove that $N(f) \subset \bigcup_{n=1}^{\infty} K_n$. For every $j \in \mathbb{N}$, the definition of $\mathscr{Y}_k$ and Proposition 4.4.10 show that

$$N(f, a_j) \subset \bigcap_{m=j}^{\infty} \bigcup_{n \in A_j^m(\boldsymbol{n}^k)} B(K_n, \delta_m) \subset \bigcup_{n=1}^{\infty} K_n.$$

It follows that

$$N(f) = \bigcup_{j=1}^{\infty} N(f, a_j) \subset \bigcup_{n=1}^{\infty} K_n.$$

∎

Lemma 4.4.15. *Let* $\boldsymbol{n} \in Z$, *and suppose that a permutation* σ *on* $\mathbb{N}$ *and* $k \in \mathbb{N}$ *satisfy* $\sigma(n) = n$ *for all* $n > n_k$. *Then we have the following:*

(1) $A_j^m(\boldsymbol{n}^k)$ *is invariant under* σ *whenever* $j \leq m$;

(2) $\sigma\big(A_j^m(\boldsymbol{n})\big) \subset A_{\max\{j,k\}}^{\max\{m,k\}}(\boldsymbol{n})$ *whenever* $j \leq m$.

Proof. Note that every subset of $\mathbb{N}$ that contains $[n_k]$ is invariant under σ.

(1) The assertion follows from the observation that

$$A_j^m(\boldsymbol{n}^k) \supset [n_j^k] = [n_{j+k}] \supset [n_k].$$

(2) If $k \leq j$, then $A_j^m(\boldsymbol{n}) \supset [n_j] \supset [n_k]$ and so $\sigma\big(A_j^m(\boldsymbol{n})\big) = A_j^m(\boldsymbol{n})$. If $j < k \leq m$, then $\sigma\big(A_j^m(\boldsymbol{n})\big) \subset \sigma\big(A_k^m(\boldsymbol{n})\big) = A_k^m(\boldsymbol{n})$. If $m < k$, then $\sigma\big(A_j^m(\boldsymbol{n})\big) \subset \sigma([n_m]) \subset \sigma([n_k]) = [n_k] = A_k^k(\boldsymbol{n})$. ∎

Proposition 4.4.16. *If $f \in C(I)$, then $\{\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \mid (\boldsymbol{K}, f) \in \mathscr{X}\}$ is invariant under finite permutations.*

Proof. Suppose that $\boldsymbol{K}$ belongs to the set and that σ is a permutation on $\mathbb{N}$ that fixes all but finitely many positive integers. Define $\boldsymbol{K}' \in \mathcal{K}^{\mathbb{N}}$ by setting $K'_n = K_{\sigma(n)}$ for $n \in \mathbb{N}$. We need to prove that $(\boldsymbol{K}', f) \in \mathscr{X}$.

Take $\boldsymbol{n} \in Z$, $\boldsymbol{\delta} \in Y$, $\boldsymbol{a}, \boldsymbol{b} \in X$, and $k \in \mathbb{Z}_+$ so that $(\boldsymbol{K}, f, \boldsymbol{n}, \boldsymbol{\delta}, \boldsymbol{a}, \boldsymbol{b}) \in \mathscr{Y}_k$. By Proposition 4.4.13, we may assume that k is so large that $\sigma(n) = n$ for all $n > n_k$.

By Lemma 4.4.15 (1), it is easy to see that $\boldsymbol{K}' \in \mathscr{S}_k(\boldsymbol{n}, \boldsymbol{\delta})$ and that $N(f, a_j) \subset \bigcup_{n \in A_j^m(\boldsymbol{n}^k)} B(K'_n, \delta_m)$ whenever $j \leq m$.

Now define $\boldsymbol{b}' \in X$ by setting $b'_j = b_{j+k}$ for $j \in \mathbb{N}$. Then for $j, m \in \mathbb{N}$ with $j \leq m$, Lemma 4.4.15 (2) shows that

$$\begin{aligned}\bigcup_{n \in A_j^m(\boldsymbol{n})} K'_n \subset \bigcup_{n \in A_{\max\{j,k\}}^{\max\{m,k\}}(\boldsymbol{n})} K_n &\subset B\big(N(f, b_{\max\{j,k\}}), \delta_{\max\{m,k\}}\big) \\ &\subset B\big(N(f, b'_j), \delta_m\big)\end{aligned}$$

because $b_{\max\{j,k\}} \leq b_{j+k} = b'_j$ and $\delta_{\max\{m,k\}} \leq \delta_m$.

Hence we have shown that $(\boldsymbol{K}', f, \boldsymbol{n}, \boldsymbol{\delta}, \boldsymbol{a}, \boldsymbol{b}') \in \mathscr{Y}_k$, from which it follows that $(\boldsymbol{K}', f) \in \mathscr{X}$. ∎

Proposition 4.4.17. *The set $\mathscr{X}$ is an analytic subset of $\mathcal{K}^{\mathbb{N}} \times C(I)$.*

Remark 4.4.18. For the following proof, tilde ˜ does not have its usual meaning and is not related to hat ˆ or check ˇ in the usual way.

Proof (*of Proposition* 4.4.17). Let $\operatorname{pr}\colon \mathcal{K}^{\mathbb{N}}\times C(I)\times Z\times Y\times X\times X \longrightarrow \mathcal{K}^{\mathbb{N}}\times C(I)$ be the projection. It suffices to prove that $\operatorname{pr}\mathscr{Y}_k = \operatorname{pr}\overline{\mathscr{Y}}_k$ for every $k\in\mathbb{Z}_+$, because it will imply that

$$\mathscr{X} = \operatorname{pr}\mathscr{Y} = \operatorname{pr}\left(\bigcup_{k=0}^{\infty}\mathscr{Y}_k\right) = \bigcup_{k=0}^{\infty}\operatorname{pr}\mathscr{Y}_k = \bigcup_{k=0}^{\infty}\operatorname{pr}\overline{\mathscr{Y}}_k,$$

from which it follows that $\mathscr{X}$ is analytic.

Let $k\in\mathbb{Z}_+$. We only need to prove that $\operatorname{pr}\overline{\mathscr{Y}}_k\subset\operatorname{pr}\mathscr{Y}_k$, so let $(\boldsymbol{K},f)\in\operatorname{pr}\overline{\mathscr{Y}}_k$ be given. Take $\boldsymbol{n}\in Z$, $\boldsymbol{\delta}\in Y$, $\boldsymbol{a},\boldsymbol{b}\in X$ with $(\boldsymbol{K},f,\boldsymbol{n},\boldsymbol{\delta},\boldsymbol{a},\boldsymbol{b})\in\overline{\mathscr{Y}}_k$. Choosing $\boldsymbol{\delta}'\in Y$, $\boldsymbol{a}',\boldsymbol{b}'\in X$ so that $\delta'_j>\delta_j$, $a'_j<a_j$, $b'_j>b_j$ for all $j\in\mathbb{N}$, we shall show that $(\boldsymbol{K},f,\boldsymbol{n},\boldsymbol{\delta}',\boldsymbol{a}',\boldsymbol{b}')\in\mathscr{Y}_k$; it will imply that $(\boldsymbol{K},f)\in\operatorname{pr}\mathscr{Y}_k$, completing the proof.

Firstly, we show that $\boldsymbol{K}\in\mathscr{S}_k(\boldsymbol{n},\boldsymbol{\delta}')$. Fix any $j_0,m_0\in\mathbb{N}$ with $2\le j_0\le m_0-1$. Take $\varepsilon>0$ with $\varepsilon<\delta'_{m_0}-\delta_{m_0}$. Since $(\boldsymbol{K},f,\boldsymbol{n},\boldsymbol{\delta},\boldsymbol{a},\boldsymbol{b})\in\overline{\mathscr{Y}}_k$, we may find $(\tilde{\boldsymbol{K}},\tilde{f},\tilde{\boldsymbol{n}},\tilde{\boldsymbol{\delta}},\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}})\in\mathscr{Y}_k$ such that

- $d(\tilde{K}_n,K_n)<\varepsilon/2$ for $n\in[n_{m_0+k}]$;
- $\tilde{n}_j=n_j$ for $j\in[m_0+k]$;
- $\varepsilon<\delta'_{m_0}-\tilde{\delta}_{m_0}$.

We write $A^m_j=A^m_j(\boldsymbol{n}^k)$ and $\tilde{A}^m_j=A^m_j(\tilde{\boldsymbol{n}}^k)$ for simplicity. Observe that

$$\tilde{A}^{m_0}_{j_0}\setminus\tilde{A}^{m_0-1}_{j_0}=A^{m_0}_{j_0}\setminus A^{m_0-1}_{j_0},\qquad \tilde{A}^{m_0-1}_{j_0-1}=A^{m_0-1}_{j_0-1},$$

and that if n belongs to either of these sets, then $d(\tilde{K}_n,K_n)<\varepsilon/2$. Accordingly, we have

$$\begin{aligned}\bigcup_{n\in A_{j_0}^{m_0}\setminus A_{j_0}^{m_0-1}} K_n \subset \bigcup_{n\in A_{j_0}^{m_0}\setminus A_{j_0}^{m_0-1}} B(\tilde{K}_n,\varepsilon/2) &= \bigcup_{n\in \tilde{A}_{j_0}^{m_0}\setminus \tilde{A}_{j_0}^{m_0-1}} B(\tilde{K}_n,\varepsilon/2)\\ &\subset \bigcup_{n\in \tilde{A}_{j_0-1}^{m_0-1}} B(\tilde{K}_n,\tilde{\delta}_{m_0}+\varepsilon/2) = \bigcup_{n\in A_{j_0-1}^{m_0-1}} B(\tilde{K}_n,\tilde{\delta}_{m_0}+\varepsilon/2)\\ &\subset \bigcup_{n\in A_{j_0-1}^{m_0-1}} B(K_n,\tilde{\delta}_{m_0}+\varepsilon) \subset \bigcup_{n\in A_{j_0-1}^{m_0-1}} B(K_n,\delta'_{m_0}).\end{aligned}$$

Hence we obtain $\boldsymbol{K}\in\mathscr{S}_k(\boldsymbol{n},\boldsymbol{\delta}')$.

Now what remains to be shown is that if $j_0\le m_0$, then

$$N(f,a'_{j_0})\subset \bigcup_{n\in A_{j_0}^{m_0}(\boldsymbol{n}^k)} B(K_n,\delta'_{m_0}), \qquad \bigcup_{n\in A_{j_0}^{m_0}(\boldsymbol{n})} K_n \subset B\big(N(f,b'_{j_0}),\delta'_{m_0}\big).$$

Fix such j_0 and m_0, and take $\varepsilon>0$ with $\varepsilon<\delta'_{m_0}-\delta_{m_0}$. Since $(\boldsymbol{K},f,\boldsymbol{n},\boldsymbol{\delta},\boldsymbol{a},\boldsymbol{b})\in\overline{\mathscr{Y}}_k$, we may find $(\tilde{\boldsymbol{K}},\tilde{f},\tilde{\boldsymbol{n}},\tilde{\boldsymbol{\delta}},\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}})\in\mathscr{Y}_k$ such that

- $d(\tilde{K}_n,K_n)<\varepsilon/2$ for $n\in[n_{m_0+k}]$;
- $\tilde{n}_j=n_j$ for $j\in[m_0+k]$;
- $\varepsilon<\delta'_{m_0}-\tilde{\delta}_{m_0}$;
- $a'_{j_0}<\tilde{a}_{j_0}$ and $b'_{j_0}>\tilde{b}_{j_0}$;
- $N(f,a'_{j_0})\subset B\big(N(\tilde{f},\tilde{a}_{j_0}),\varepsilon/2\big)$ and $N(\tilde{f},\tilde{b}_{j_0})\subset B\big(N(f,b'_{j_0}),\varepsilon/2\big)$, which can be established because of Proposition 4.3.8.

Observe that

$$A_{j_0}^{m_0}(\tilde{\boldsymbol{n}}^k)=A_{j_0}^{m_0}(\boldsymbol{n}^k), \qquad A_{j_0}^{m_0}(\tilde{\boldsymbol{n}})=A_{j_0}^{m_0}(\boldsymbol{n}),$$

and that if n belongs to either of these sets, then $d(\tilde{K}_n,K_n)<\varepsilon/2$. Accordingly, we have

$$\begin{aligned} N(f, a'_{j_0}) &\subset B\big(N(\tilde{f}, \tilde{a}_{j_0}), \varepsilon/2\big) \subset \bigcup_{n \in A_{j_0}^{m_0}(\tilde{\boldsymbol{n}}^k)} B(\tilde{K}_n, \tilde{\delta}_{m_0} + \varepsilon/2) \\ &= \bigcup_{n \in A_{j_0}^{m_0}(\boldsymbol{n}^k)} B(\tilde{K}_n, \tilde{\delta}_{m_0} + \varepsilon/2) \subset \bigcup_{n \in A_{j_0}^{m_0}(\boldsymbol{n}^k)} B(K_n, \tilde{\delta}_{m_0} + \varepsilon) \\ &\subset \bigcup_{n \in A_{j_0}^{m_0}(\boldsymbol{n}^k)} B(K_n, \delta'_{m_0}) \end{aligned}$$

and

$$\begin{aligned} \bigcup_{n \in A_{j_0}^{m_0}(\boldsymbol{n})} K_n &\subset \bigcup_{n \in A_{j_0}^{m_0}(\boldsymbol{n})} B(\tilde{K}_n, \varepsilon/2) = \bigcup_{n \in A_{j_0}^{m_0}(\tilde{\boldsymbol{n}})} B(\tilde{K}_n, \varepsilon/2) \\ &\subset B\big(N(\tilde{f}, \tilde{b}_{j_0}), \tilde{\delta}_{m_0} + \varepsilon/2\big) \subset B\big(N(f, b'_{j_0}), \tilde{\delta}_{m_0} + \varepsilon\big) \\ &\subset B\big(N(f, b'_{j_0}), \delta'_{m_0}\big). \end{aligned}$$

∎

4.4.3 Key Proposition

We reduce the main theorem (Theorem 4.1.2 or equivalently Theorem 4.3.7) to a proposition, which we shall refer to as *Key Proposition*.

Proposition 4.4.19 (Key Proposition). *If $\mathscr{A}$ is a residual subset of $\mathcal{K}^{\mathbb{N}}$, then a typical function $f \in C(I)$ has the property that $(\boldsymbol{K}, f) \in \mathscr{X}$ for some $\boldsymbol{K} \in \mathscr{A}$.*

The proof of the key proposition will be given in the next section; here we only show that it implies the main theorem.

Proposition 4.4.20. *The key proposition implies the main theorem. That is to say, if the key proposition is true, then a subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$ is residual if and only if $N(f) \in \mathcal{F}$ for a typical function $f \in C(I)$.*

Proof. Suppose first that $\mathcal{F}$ is residual. Then the key proposition applied to $\mathscr{A} = \{\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ tells us that a typical function $f \in C(I)$ has the property that $(\boldsymbol{K}, f) \in \mathscr{X}$ for some $\boldsymbol{K} \in \mathscr{A}$, which implies that $N(f) = \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$ by Proposition 4.4.14.

Conversely, suppose that a typical function $f \in C(I)$ has the property that $N(f) \in \mathcal{F}$. Then we may take a dense G_δ subset G of $C(I)$ contained in $\{f \in C(I) \mid N(f) \in \mathcal{F}\}$. Write $\mathscr{A}$ for the set of all $\boldsymbol{K} \in \mathcal{K}^{\mathbb{N}}$ such that $(\boldsymbol{K}, f) \in \mathscr{X}$ for some $f \in G$. Observe that $\mathscr{A}$ is invariant under finite permutations because it is a union of sets invariant under finite permutations by Proposition 4.4.16. Since $\mathscr{A}$ is the projection of $\mathscr{X} \cap (\mathcal{K}^{\mathbb{N}} \times G)$ to $\mathcal{K}^{\mathbb{N}}$, Propositions 2.3.5 and 4.4.17 show that $\mathscr{A}$ is analytic. It follows from Theorem 2.3.8 that $\mathscr{A}$ has the Baire property. Therefore Proposition 4.4.4 implies that $\mathscr{A}$ is either meagre or residual. If $\mathscr{A}$ is meagre, then the key proposition applied to $\mathscr{A}^c$ and the residuality of G imply that $(\boldsymbol{K}, f) \in \mathscr{X}$ for some $f \in G$ and $\boldsymbol{K} \in \mathscr{A}^c$, which contradicts the definition of $\mathscr{A}$. Hence $\mathscr{A}$ is residual. This completes the proof because if $\boldsymbol{K} \in \mathscr{A}$, then for some $f \in G$ we have $\bigcup_{n=1}^{\infty} K_n = N(f) \in \mathcal{F}$ by Proposition 4.4.14. ∎

In terms of the Banach-Mazur game, we can rephrase the key proposition in the following form:

Proposition 4.4.21 (Key Proposition). *If $\mathscr{A}$ is a residual subset of $\mathcal{K}^{\mathbb{N}}$ and*

$$S = \{f \in C(I) \mid (\boldsymbol{K}, f) \in \mathscr{X} \text{ for some } \boldsymbol{K} \in \mathscr{A}\},$$

then the $\big(C(I), S, \mathcal{B}\big)$-Banach-Mazur game admits a winning strategy for Player II, where $\mathcal{B}$ is the family of all open balls in $C(I)$ whose centres are C^1 functions.

4.5 Proof of the key proposition

This section will be devoted to the proof of the key proposition (Proposition 4.4.21). The reader may wish to look at the outline of the proof in Section 4.6 before reading the detailed proof in the present section.

4.5.1 Introduction to the strategy

Let $\mathscr{A}$ be a residual subset of $\mathcal{K}^{\mathbb{N}}$, and take open dense subsets $\mathscr{U}_m$ of $\mathcal{K}^{\mathbb{N}}$ for $m \in \mathbb{N}$ so that $\bigcap_{m=1}^{\infty} \mathscr{U}_m \subset \mathscr{A}$. Define S as in Proposition 4.4.21.

We shall use two sequences of positive numbers a_j and b_j, and their cousins $a_j^{m,k}$ and $b_j^{m,k}$. The numbers a_j are defined by $a_j = j$ for $j \in \mathbb{N}$, and the numbers $a_j^{m,k}$, where $j \leq m$ and $k \in [4]$, are chosen to satisfy

$$\begin{aligned} a_{j+1} = j+1 > a_j^{j,1} > a_j^{j,2} > a_j^{j,3} > a_j^{j,4} \\ = a_j^{j+1,1} > a_j^{j+1,2} > a_j^{j+1,3} > a_j^{j+1,4} \\ = \cdots \\ \to a_j = j \end{aligned}$$

(for example, $a_j^{m,k} = j + 2^{-(3m+k)}$). The numbers b_j are defined in the strategy, each b_j being determined in the jth round, and they satisfy $b_j < b_{j+1}$ and $b_j > j+2$ for all $j \in \mathbb{N}$. As soon as each b_j is determined, the numbers $b_j^{m,k}$ for $m \geq j$ and $k \in [3]$ are chosen to satisfy

$$\begin{aligned} j+1 < b_j - 1 < b_j^{j,1} < b_j^{j,2} < b_j^{j,3} \\ = b_j^{j+1,1} < b_j^{j+1,2} < b_j^{j+1,3} \\ = \cdots \\ \to b_j \end{aligned}$$

(for example, $b_j^{m,k} = b_j - 2^{-(2m+k)}$). Note that $a_j^{m,k} < j+1 < b_j^{m',k'}$ for all j, m, m', k, k'.

The moves of Players I and II in the mth round will be denoted by $B(f_m, \alpha_m)$ and $B(g_m, \beta_m)$ respectively. By the rule of the game, the functions f_m and g_m are all continuously differentiable. In the mth round, Player II will construct, in addition to g_m and β_m, the following: a positive number h_m, a positive number μ_m, finite subsets $\tilde{L}_n^m$ of I, a sequence $\boldsymbol{K}^m \in \mathcal{K}^{\mathbb{N}}$ (and its partition $K_n^m = \hat{K}_n^m \amalg \check{K}_n^m$), a positive integer n_m, a positive number w_m, and a positive number b_m (as mentioned above). They will be chosen to satisfy a number of properties, but the following, written as $(\bigstar_m)$ afterwards, is essential to ensure that the induction proceeds: if $f \in B(g_m, \beta_m)$, then

- $\tilde{N}(f, a_j^{m,4}) \subset \bigcup_{n \in A_j^m} B(\tilde{K}_n^m, w_m)$,
- $\bigcup_{n \in A_j^m} \tilde{K}_n^m \subset B\big(\tilde{N}(f, b_j^{m,3}), w_m\big)$,
- $\tilde{N}(f, a_j^{m,4}) \cap \bigcup_{n \in [n_m] \setminus A_j^m} \overline{B}(\tilde{K}_n^m, w_m) = \emptyset$

for every $j \in [m]$. Here $A_j^m = A_j^m(\boldsymbol{n})$, where $\boldsymbol{n} = (n_m)$ is the sequence of positive integers whose mth term will be defined in the mth round by Player II. We must be careful exactly when A_j^m will be determined; it is true that the whole sequence $\boldsymbol{n}$ will be determined only after the game is over, but since A_j^m depends only on n_k for $k \in [\max\{j, m-1\}]$, we can use A_j^m once $n_{\max\{j,m-1\}}$ is determined.

4.5.2 First round

Suppose that Player I has given his first move $B(f_1, \alpha_1)$.

Construction of h_1, μ_1, L_n^1, $\boldsymbol{K}^1$, n_1, and w_1

Take $h_1 > 0$ with $h_1 < \alpha_1$, and set $\mu_1 = \mu(f_1, a_1^{1,3}, a_1^{1,2}, h_1)$ (recall Definition 4.3.18). Put $\tilde{L}_n^1 = I$ for every $n \in \mathbb{N}$. There exists $\boldsymbol{K}^1 \in \mathscr{U}_1 \cap \mathscr{D}$ such that we may partition K_1^1 as $K_1^1 = \hat{K}_1^1 \amalg \check{K}_1^1$ in such a way that $B(\tilde{K}_1^1, \mu_1) = I$. Choose $n_1 \in \mathbb{N}$ and $w_1 > 0$ so that $\overline{U}(\boldsymbol{K}^1, n_1, 2w_1) \subset \mathscr{U}_1$; make w_1 smaller, if necessary, so that the balls $\overline{B}(x, w_1)$ for $x \in \bigcup_{n=1}^{n_1} K_n^1$ are disjoint.

Construction of g_1 and b_1

Let φ_1 be a bump function of height h_1 and width w_1 located at $\hat{K}_1^1$ and $\check{K}_1^1$. Define $g_1 = f_1 + \varphi_1$. It is clear that $g_1 \in B(f_1, \alpha_1)$. Since $\mu_1 = \mu(f_1, a_1^{1,3}, a_1^{1,2}, h_1)$ and $B(\tilde{K}_1^1, \mu_1) = I$, Proposition 4.3.17 shows that

$$\tilde{N}(g_1, a_1^{1,3}) \subset \tilde{N}(f_1, a_1^{1,2}) \cap B(\tilde{K}_1^1, w_1) \subset B(\tilde{K}_1^1, w_1) \subset \bigcup_{n=1}^{n_1} B(\tilde{K}_n^1, w_1).$$

Let $b_1 > 3$ be so large that $b_1^{1,2} \geq \|g_1'\|$. Then $\tilde{N}(g_1, b_1^{1,2}) = I \supset \bigcup_{n=1}^{n_1} \tilde{K}_n^1$. Since $A_1^1 = [n_1]$, we have

- $\tilde{N}(g_1, a_1^{1,3}) \subset \bigcup_{n \in A_1^1} B(\tilde{K}_n^1, w_1)$;
- $\bigcup_{n \in A_1^1} \tilde{K}_n^1 \subset \tilde{N}(g_1, b_1^{1,2})$.

Construction of β_1

We may find $\varepsilon_1 > 0$ such that

- $\tilde{N}(g_1, a_1^{1,3}) \subset \bigcup_{n \in A_1^1} B(\tilde{K}_n^1, w_1 - \varepsilon_1)$.

By Proposition 4.3.8, there exists $\beta_1 > 0$ with $B(g_1, \beta_1) \subset B(f_1, \alpha_1)$ such that whenever $f \in B(g_1, \beta_1)$, we have

- $\tilde{N}(f, a_1^{1,4}) \subset B\big(\tilde{N}(g_1, a_1^{1,3}), \varepsilon_1\big)$;
- $\tilde{N}(g_1, b_1^{1,2}) \subset B\big(\tilde{N}(f, b_1^{1,3}), w_1\big)$.

It follows that whenever $f \in B(g_1, \beta_1)$, we have

- $\tilde{N}(f, a_1^{1,4}) \subset \bigcup_{n \in A_1^1} B(\tilde{K}_n^1, w_1)$;
- $\bigcup_{n \in A_1^1} \tilde{K}_n^1 \subset B\big(\tilde{N}(f, b_1^{1,3}), w_1\big)$;
- $\tilde{N}(f, a_1^{1,4}) \cap \bigcup_{n \in [n_1] \setminus A_1^1} \overline{B}(\tilde{K}_n^1, w_1) = \emptyset$,

the last condition being trivial because $[n_1] \setminus A_1^1 = \emptyset$. Therefore ($\bigstar_1$) has been established.

4.5.3 mth round for $m \geq 2$

Let $m \geq 2$ and suppose that Player I has given his mth move $B(f_m, \alpha_m)$. Since the rule of the Banach-Mazur game requires that $f_m \in B(g_{m-1}, \beta_{m-1})$, it follows from ($\bigstar_{m-1}$) that

- $\tilde{N}(f_m, a_j^{m,1}) \subset \bigcup_{n \in A_j^{m-1}} B(\tilde{K}_n^{m-1}, w_{m-1})$,
- $\bigcup_{n \in A_j^{m-1}} \tilde{K}_n^{m-1} \subset B\big(\tilde{N}(f_m, b_j^{m,1}), w_{m-1}\big)$,

- $\tilde{N}(f_m, a_j^{m,1}) \cap \bigcup_{n\in[n_{m-1}]\setminus A_j^{m-1}} \overline{B}(\tilde{K}_n^{m-1}, w_{m-1}) = \emptyset$

for every $j \in [m-1]$ (remember that $a_j^{m-1,4} = a_j^{m,1}$ and $b_j^{m-1,3} = b_j^{m,1}$).

Construction of h_m and μ_m

Take $h_m > 0$ with $h_m < \alpha_m$, and set

$$\mu_m = \min_{j\in[m]} \mu(f_m, a_j^{m,3}, a_j^{m,2}, h_m) > 0.$$

Construction of L_n^m

Choosing an auxiliary number $\zeta_m > 0$ so that

- $\tilde{N}(f_m, a_j^{m,1}) \subset \bigcup_{n\in A_j^{m-1}} B(\tilde{K}_n^{m-1}, w_{m-1} - \zeta_m)$ for $j \in [m-1]$,

we shall define finite subsets $\tilde{L}_n^m$ of I for $n \in [n_{m-1} + m - 1]$.

Firstly, let $n \in [n_{m-1}]$ and take the minimum $j \in [m-1]$ with $n \in A_j^{m-1}$. When x varies in $\tilde{N}(f_m, b_j^{m,1}) \cap B(\tilde{K}_n^{m-1}, w_{m-1})$,

- the open balls $B(x, w_{m-1})$ cover $\tilde{K}_n^{m-1}$;
- the open balls $B(x, \mu_m)$ cover $\tilde{N}(f_m, a_j^{m,1}) \cap \overline{B}(\tilde{K}_n^{m-1}, w_{m-1} - \zeta_m)$.

The compactness of the sets covered gives us a finite subset $\tilde{L}_n^m$ of $\tilde{N}(f_m, b_j^{m,1}) \cap B(\tilde{K}_n^{m-1}, w_{m-1})$ such that

- $B(\tilde{L}_n^m, w_{m-1}) \supset \tilde{K}_n^{m-1}$;
- $B(\tilde{L}_n^m, \mu_m) \supset \tilde{N}(f_m, a_j^{m,1}) \cap \overline{B}(\tilde{K}_n^{m-1}, w_{m-1} - \zeta_m)$.

Secondly, for $j \in [m-1] \setminus \{1\}$, we set

$$\tilde{P}_j^m = \big(\tilde{N}(f_m, a_j^{m,1}) \setminus \operatorname{Int} \tilde{N}(f_m, a_{j-1}^{m,1})\big) \cap \bigcup_{n\in A_{j-1}^{m-1}} \overline{B}(\tilde{K}_n^{m-1}, w_{m-1} - \zeta_m),$$

and define $\tilde{L}_{n_{m-1}+j-1}^m$ as a finite subset of $\tilde{P}_j^m$ such that $B(\tilde{L}_{n_{m-1}+j-1}^m, \mu_m) \supset \tilde{P}_j^m$. This defines $\tilde{L}_n^m$ for $n \in [n_{m-1} + m - 2] \setminus [n_{m-1}]$.

Lastly, we define $\tilde{L}^m_{n_{m-1}+m-1}$ as a finite subset of $I \setminus \bigcup_{n=1}^{n_{m-1}+m-2} B(\tilde{L}^m_n, \mu_m)$ such that $B(\tilde{L}^m_{n_{m-1}+m-1}, \mu_m) \supset I \setminus \bigcup_{n=1}^{n_{m-1}+m-2} B(\tilde{L}^m_n, \mu_m)$.

Having defined $\tilde{L}^m_n$ for $n \in [n_{m-1} + m - 1]$, we prove the following claim. Remember that since $n_1, \ldots, n_{m-1}$ have already been defined, we know A^m_j for $j \in [m-1]$.

Claim 4. *We have the following*:

(1) $d(\tilde{L}^m_n, \tilde{K}^{m-1}_n) < w_{m-1}$ *for* $n \in [n_{m-1}]$;

(2) $\tilde{N}(f_m, a^{m,1}_{m-1}) \subset \bigcup_{n=1}^{n_{m-1}+m-2} B(\tilde{L}^m_n, \mu_m)$;

(3) $\bigcup_{n \in A^m_j} \tilde{L}^m_n \subset \tilde{N}(f_m, b^{m,1}_j)$ *for* $j \in [m-1]$;

(4) $\bigcup_{n=1}^{n_{m-1}+m-1} B(\tilde{L}^m_n, \mu_m) = I$;

(5) $\bigcup_{n \in A^m_j \setminus A^{m-1}_j} \tilde{L}^m_n \subset \bigcup_{n \in A^{m-1}_{j-1}} B(\tilde{L}^m_n, 2w_{m-1})$ *for* $j \in [m-1] \setminus \{1\}$;

(6) $\tilde{N}(f_m, a^{m,2}_j) \cap \bigcup_{n \in [n_{m-1}+m-1] \setminus A^m_j} \tilde{L}^m_n = \emptyset$ *for* $j \in [m-1]$.

Proof. (1) Both $\tilde{L}^m_n \subset B(\tilde{K}^{m-1}_n, w_{m-1})$ and $\tilde{K}^{m-1}_n \subset B(\tilde{L}^m_n, w_{m-1})$ are clear from the definition of $\tilde{L}^m_n$.

(2) Let $x \in \tilde{N}(f_m, a^{m,1}_{m-1})$ and look at the minimum $j \in [m-1]$ with $x \in \tilde{N}(f_m, a^{m,1}_j)$.

If $j = 1$, then the definition of ζ_m tells us that $x \in B(\tilde{K}^{m-1}_n, w_{m-1} - \zeta_m)$ for some $n \in A^{m-1}_1 = [n_1]$; for this n, the number j taken in the definition of $\tilde{L}^m_n$ must be 1, so

$$x \in \tilde{N}(f_m, a^{m,1}_1) \cap B(\tilde{K}^{m-1}_n, w_{m-1} - \zeta_m) \subset B(\tilde{L}^m_n, \mu_m).$$

Now, suppose that $j \in [m-1] \setminus \{1\}$. Since $x \in \tilde{N}(f_m, a^{m,1}_j)$, we may take $n \in A^{m-1}_j$ with $x \in B(\tilde{K}^{m-1}_n, w_{m-1} - \zeta_m)$. If $n \notin A^{m-1}_{j-1}$, then the number j taken in the definition of $\tilde{L}^m_n$ must be the same as our j, and so

$$x \in \tilde{N}(f_m, a^{m,1}_j) \cap B(\tilde{K}^{m-1}_n, w_{m-1} - \zeta_m) \subset B(\tilde{L}^m_n, \mu_m).$$

If $n \in A_{j-1}^{m-1}$, then $x \in \tilde{P}_j^m$ because $x \notin \tilde{N}(f_m, a_{j-1}^{m,1}) \supset \operatorname{Int} \tilde{N}(f_m, a_{j-1}^{m,1})$ by the minimality of j; therefore $x \in B(\tilde{L}^m_{n_{m-1}+j-1}, \mu_m)$, which implies the required inclusion.

(3) Let $x \in \bigcup_{n \in A_j^m} \tilde{L}_n^m$ and take $n \in A_j^m$ with $x \in \tilde{L}_n^m$. If $n \in A_j^{m-1}$, then taking the minimum i with $n \in A_i^{m-1}$, we have

$$x \in \tilde{L}_n^m \subset \tilde{N}(f_m, b_i^{m,1}) \subset \tilde{N}(f_m, b_j^{m,1}).$$

If $n \notin A_j^{m-1}$, then $j \geq 2$ and $n_{m-1} + 1 \leq n \leq n_{m-1} + j - 1$, from which it follows that

$$\begin{aligned} x \in \tilde{L}_n^m \subset \tilde{P}^m_{n-n_{m-1}+1} &\subset \tilde{N}(f_m, a^{m,1}_{n-n_{m-1}+1}) \\ \subset \tilde{N}(f_m, a_j^{m,1}) &\subset \tilde{N}(f_m, b_j^{m,1}). \end{aligned}$$

(4) Immediate from the definition of $\tilde{L}_{n_{m-1}+m-1}$.

(5) We have

$$\begin{aligned} \bigcup_{n \in A_j^m \setminus A_j^{m-1}} \tilde{L}_n^m &= \bigcup_{n=n_{m-1}+1}^{n_{m-1}+j-1} \tilde{L}_n^m \subset \bigcup_{k=2}^{j} \tilde{P}_k^m \\ &\subset \bigcup_{k=2}^{j} \bigcup_{n \in A_{k-1}^{m-1}} \overline{B}(\tilde{K}_n^{m-1}, w_{m-1} - \zeta_m) \\ &= \bigcup_{n \in A_{j-1}^{m-1}} \overline{B}(\tilde{K}_n^{m-1}, w_{m-1} - \zeta_m) \\ &\subset \bigcup_{n \in A_{j-1}^{m-1}} B(\tilde{L}_n^m, 2w_{m-1}), \end{aligned}$$

where the last inclusion follows from (1).

(6) We need to show that $\tilde{N}(f_m, a_j^{m,2}) \cap \tilde{L}_n^m = \emptyset$ for $n \in [n_{m-1} + m - 1] \setminus A_j^m$. There are three cases: $n \in [n_{m-1}] \setminus A_j^{m-1}$, $n_{m-1} + j \leq n \leq n_{m-1} + m - 2$, and $n = n_{m-1} + m - 1$.

If $n \in [n_{m-1}] \setminus A_j^{m-1}$, then

$$\tilde{N}(f_m, a_j^{m,2}) \cap \tilde{L}_n^m \subset \tilde{N}(f_m, a_j^{m,1}) \cap B(\tilde{K}_n^{m-1}, w_{m-1}) = \emptyset$$

by ($\bigstar_{m-1}$).

If $n_{m-1} + j \le n \le n_{m-1} + m - 2$, then

$$\begin{aligned}\tilde{N}(f_m, a_j^{m,2}) \cap \tilde{L}_n^m &\subset \tilde{N}(f_m, a_j^{m,2}) \cap \tilde{P}^m_{n-n_{m-1}+1} \\ &\subset \tilde{N}(f_m, a^{m,2}_{n-n_{m-1}}) \setminus \operatorname{Int} \tilde{N}(f_m, a^{m,1}_{n-n_{m-1}}) \\ &= \emptyset\end{aligned}$$

because of Corollary 4.3.10.

If $n = n_{m-1} + m - 1$, then (2) implies that

$$\begin{aligned}\tilde{N}(f_m, a_j^{m,2}) \cap \tilde{L}_n^m &\subset \tilde{N}(f_m, a^{m,1}_{m-1}) \cap \tilde{L}_n^m \\ &\subset \bigcup_{n'=1}^{n_{m-1}+m-2} B(\tilde{L}^m_{n'}, \mu_m) \cap \tilde{L}^m_{n_{m-1}+m-1} \\ &= \emptyset\end{aligned}$$

because of the choice of $\tilde{L}^m_{n_{m-1}+m-1}$. ∎

Construction of $\boldsymbol{K}^m$

We shall construct a sequence $\boldsymbol{K}^m \in \mathscr{U}_m \cap \mathscr{D}$ such that we may partition $K_n^m = \hat{K}_n^m \amalg \tilde{K}_n^m$ for each $n \in \mathbb{N}$ in such a way that the following conditions are fulfilled:

(1) $d(\tilde{K}_n^m, \tilde{K}_n^{m-1}) < w_{m-1}$ for $n \in [n_{m-1}]$;

(3) $\bigcup_{n \in A_j^m} \tilde{K}_n^m \subset \operatorname{Int} \tilde{N}(f_m, b_j^{m,2})$ for $j \in [m-1]$;

(4) $\bigcup_{n=1}^{n_{m-1}+m-1} B(\tilde{K}_n^m, \mu_m) = I$;

(5) $\bigcup_{n \in A_j^m \setminus A_j^{m-1}} \tilde{K}_n^m \subset \bigcup_{n \in A_{j-1}^{m-1}} B(\tilde{K}_n^m, 2w_{m-1})$ for $j \in [m-1] \setminus \{1\}$;

(6) $\tilde{N}(f_m, a_j^{m,2}) \cap \bigcup_{n \in [n_{m-1}+m-1] \setminus A_j^m} \tilde{K}_n^m = \emptyset$ for $j \in [m-1]$

(these are the relations of Claim 4 (1), (3), (4), (5), (6) with $\tilde{L}_n^m$ replaced by $\tilde{K}_n^m$ and with $\tilde{N}(f_m, b_j^{m,1})$ replaced by $\operatorname{Int} \tilde{N}(f_m, b_j^{m,2})$ in (3)).

We note that Claim 4 (3) and Corollary 4.3.10 show that $\bigcup_{n\in A_j^m}\tilde{L}_n^m \subset \operatorname{Int}\tilde{N}(f_m, b_j^{m,2})$ for $j \in [m-1]$. Therefore, by Claim 4, if we choose *disjoint* finite subsets $\hat{Q}_1^m, \ldots, \hat{Q}_{n_{m-1}+m-1}^m, \check{Q}_1^m, \ldots, \check{Q}_{n_{m-1}+m-1}^m$ of I so that the distances $d(\tilde{Q}_n^m, \tilde{L}_n^m)$ for $n \in [n_{m-1}+m-1]$ are sufficiently small, then they satisfy the following conditions:

(1) $d(\tilde{Q}_n^m, \tilde{K}_n^{m-1}) < w_{m-1}$ for $n \in [n_{m-1}]$;

(3) $\bigcup_{n\in A_j^m}\tilde{Q}_n^m \subset \operatorname{Int}\tilde{N}(f_m, b_j^{m,2})$ for $j \in [m-1]$;

(4) $\bigcup_{n=1}^{n_{m-1}+m-1} B(\tilde{Q}_n^m, \mu_m) = I$;

(5) $\bigcup_{n\in A_j^m\setminus A_j^{m-1}}\tilde{Q}_n^m \subset \bigcup_{n\in A_{j-1}^{m-1}} B(\tilde{Q}_n^m, 2w_{m-1})$ for $j \in [m-1]\setminus\{1\}$;

(6) $\tilde{N}(f_m, a_j^{m,2}) \cap \bigcup_{n\in[n_{m-1}+m-1]\setminus A_j^m}\tilde{Q}_n^m = \emptyset$ for $j \in [m-1]$.

Since $\boldsymbol{K}^m$ must belong to $\mathscr{U}_m$, we consider $\boldsymbol{K}^m \in \mathscr{U}_m \cap \mathscr{D}$ such that the distances $d(K_n^m, \hat{Q}_n^m \amalg \check{Q}_n^m)$ for $n \in [n_{m-1}+m-1]$ are so small that each point in K_n^m has the unique closest point in $\hat{Q}_n^m \amalg \check{Q}_n^m$. If the distances $d(K_n^m, \hat{Q}_n^m \amalg \check{Q}_n^m)$ are sufficiently small, the sequence $\boldsymbol{K}^m$ satisfies the required conditions.

Construction of n_m and w_m

Choose $n_m \in \mathbb{N}$ and $w_m > 0$ so that

- $n_m \geq n_{m-1} + m - 1$;
- $w_m < w_{m-1}/2$;
- $\overline{U}(\boldsymbol{K}^m, n_m, 2w_m) \subset \mathscr{U}_m$;
- $\tilde{N}(f_m, a_j^{m,2}) \cap \bigcup_{n\in[n_{m-1}+m-1]\setminus A_j^m}\overline{B}(\tilde{K}_n^m, w_m) = \emptyset$ for $j \in [m-1]$.

Make w_m smaller, if necessary, so that

- the balls $\overline{B}(x, w_m)$ for $x \in \bigcup_{n=1}^{n_m} K_n^m$ are disjoint.

Construction of g_m and b_m

Take a bump function φ_m of height h_m and width w_m located at $\bigcup_{n=1}^{n_{m-1}+m-1} \hat{K}_n^m$ and $\bigcup_{n=1}^{n_{m-1}+m-1} \check{K}_n^m$, and set $g_m = f_m + \varphi_m$.

Let $b_m > \max\{m+2, b_{m-1}\}$ be so large that $b_m^{m,2} \geq \|g_m'\|$.

Claim 5. *(1) $\tilde{N}(g_m, a_j^{m,3}) \subset \bigcup_{n\in A_j^m} B(\tilde{K}_n^m, w_m)$ for $j \in [m]$.*

(2) $\bigcup_{n\in A_j^m} \tilde{K}_n^m \subset \tilde{N}(g_m, b_j^{m,2})$ for $j \in [m]$.

(3) $\tilde{N}(g_m, a_j^{m,3}) \cap \bigcup_{n\in[n_m]\setminus A_j^m} \overline{B}(\tilde{K}_n^m, w_m) = \emptyset$ for $j \in [m]$.

Proof. (1) Remember the definition of μ_m and property (4) of $\boldsymbol{K}^m$. If $j = m$, then $A_j^m = A_m^m = [n_m]$ and

$$\begin{aligned}\tilde{N}(g_m, a_m^{m,3}) &\subset \tilde{N}(f_m, a_m^{m,2}) \cap \bigcup_{n=1}^{n_{m-1}+m-1} B(\tilde{K}_n^m, w_m)\\ &\subset \bigcup_{n=1}^{n_{m-1}+m-1} B(\tilde{K}_n^m, w_m) \subset \bigcup_{n=1}^{n_m} B(\tilde{K}_n^m, w_m).\end{aligned}$$

If $j \in [m-1]$, then the choice of w_m implies that

$$\begin{aligned}\tilde{N}(g_m, a_j^{m,3}) &\subset \tilde{N}(f_m, a_j^{m,2}) \cap \bigcup_{n=1}^{n_{m-1}+m-1} B(\tilde{K}_n^m, w_m)\\ &\subset \bigcup_{n\in A_j^m} B(\tilde{K}_n^m, w_m).\end{aligned}$$

(2) If $j = m$, then the choice of b_m implies that

$$\tilde{N}(g_m, b_j^{m,2}) = \tilde{N}(g_m, b_m^{m,2}) = I \supset \bigcup_{n\in A_j^m} \tilde{K}_n^m.$$

If $j \in [m-1]$, then property (3) of $\boldsymbol{K}^m$ and Proposition 4.3.16 show that

$$\bigcup_{n\in A_j^m} \tilde{K}_n^m \subset \bigcup_{n=1}^{n_{m-1}+m-1} \tilde{K}_n^m \cap \tilde{N}(f_m, b_j^{m,2}) \subset \tilde{N}(g_m, b_j^{m,2}).$$

(3) If $j = m$, then the claim is trivial because $[n_m] \setminus A_j^m = \emptyset$. If $j \in [m-1]$, then (1) and the choice of w_m show that

$$\begin{aligned}&\tilde{N}(g_m, a_j^{m,3}) \cap \bigcup_{n \in [n_m] \setminus A_j^m} \overline{B}(\tilde{K}_n^m, w_m) \\ &\subset \bigcup_{n \in A_j^m} B(\tilde{K}_n^m, w_m) \cap \bigcup_{n \in [n_m] \setminus A_j^m} \overline{B}(\tilde{K}_n^m, w_m) = \emptyset.\end{aligned}$$

∎

Construction of β_m

We choose $\beta_m > 0$ as in the following claim:

Claim 6. *There exists $\beta_m > 0$ with $B(g_m, \beta_m) \subset B(f_m, \alpha_m)$ such that if $f \in B(g_m, \beta_m)$, then*

- $\tilde{N}(f, a_j^{m,4}) \subset \bigcup_{n \in A_j^m} B(\tilde{K}_n^m, w_m)$,
- $\bigcup_{n \in A_j^m} \tilde{K}_n^m \subset B\big(\tilde{N}(f, b_j^{m,3}), w_m\big)$,
- $\tilde{N}(f, a_j^{m,4}) \cap \bigcup_{n \in [n_m] \setminus A_j^m} \overline{B}(\tilde{K}_n^m, w_m) = \emptyset$

for every $j \in [m]$.

Proof. By Claim 5, we may find $\varepsilon_m > 0$ such that

- $\tilde{N}(g_m, a_j^{m,3}) \subset \bigcup_{n \in A_j^m} B(\tilde{K}_n^m, w_m - \varepsilon_m)$,
- $\bigcup_{n \in A_j^m} \tilde{K}_n^m \subset \tilde{N}(g_m, b_j^{m,2})$,
- $B\big(\tilde{N}(g_m, a_j^{m,3}), \varepsilon_m\big) \cap \bigcup_{n \in [n_m] \setminus A_j^m} \overline{B}(\tilde{K}_n^m, w_m) = \emptyset$

for every $j \in [m]$ (note that there is no ε_m in the second condition). By Proposition 4.3.8, there exists $\beta_m > 0$ with $B(g_m, \beta_m) \subset B(f_m, \alpha_m)$ such that if $f \in B(g_m, \beta_m)$, then

- $\tilde{N}(f, a_j^{m,4}) \subset B\big(\tilde{N}(g_m, a_j^{m,3}), \varepsilon_m\big)$,
- $\tilde{N}(g_m, b_j^{m,2}) \subset B\big(\tilde{N}(f, b_j^{m,3}), w_m\big)$

for every $j \in [m]$. It is easy to see that this β_m satisfies the required condition. ∎

4.5.4 Proof that the strategy makes Player II win

Proposition 4.5.1. (1) *For every $n \in \mathbb{N}$, the sequence $(K_n^m)_{m\in\mathbb{N}}$ converges in $\mathcal{K}$. Denote the limit by K_n.*

(2) *We have $d(K_n, K_n^m) \le 2w_m$ whenever $n \in [n_m]$.*

(3) *The sequence $\boldsymbol{K} = (K_n)_{n\in\mathbb{N}}$ belongs to $\mathscr{A}$.*

Proof. Remember the following:

- if $n \in [n_m]$, then $d(K_n^{m+1}, K_n^m) < w_m$ because $d(\tilde{K}_n^{m+1}, \tilde{K}_n^m) < w_m$;
- $w_{m+1} < w_m/2$ and $\overline{U}(\boldsymbol{K}^m, n_m, 2w_m) \subset \mathscr{U}_m$.

(1) Fix $n \in \mathbb{N}$ and denote by m_0 the least positive integer with $n \in [n_{m_0}]$. Then, since $d(K_n^{m+1}, K_n^m) < w_m$ for all $m \ge m_0$, we have, for all m and m' with $m_0 \le m < m'$,

$$d(K_n^{m'}, K_n^m) \le \sum_{k=m}^{m'-1} d(K_n^{k+1}, K_n^k) < \sum_{k=m}^{m'-1} w_k \le \sum_{k=m}^{m'-1} 2^{-(k-m)} w_m < 2w_m.$$

It follows that $(K_n^m)_{m\in\mathbb{N}}$ is a Cauchy sequence and therefore converges.

(2) Obvious from the estimate in the proof of (1).

(3) It follows from (2) that

$$\boldsymbol{K} \in \bigcap_{m=1}^{\infty} \overline{U}(\boldsymbol{K}^m, n_m, 2w_m) \subset \bigcap_{m=1}^{\infty} \mathscr{U}_m \subset \mathscr{A}.$$

∎

Proposition 4.5.2. *If $f \in \bigcap_{m=1}^{\infty} B(g_m, \beta_m)$, then*

$$N(f, a_j) \subset \bigcup_{n\in A_j^m} B(K_n, 3w_m) \quad \text{and} \quad \bigcup_{n\in A_j^m} K_n \subset B\big(N(f, b_j), 3w_m\big)$$

whenever $j \le m$.

Proof. Suppose that $j \le m$. Then by the choice of β_m (Claim 6), we have

- $\tilde{N}(f, a_j^{m,4}) \subset \bigcup_{n\in A_j^m} B(\tilde{K}_n^m, w_m)$;

- $\bigcup_{n\in A_j^m} \tilde{K}_n^m \subset B\big(\tilde{N}(f, b_j^{m,3}), w_m\big)$.

Taking the union for ˆ and ˇ gives

$$N(f, a_j^{m,4}) \subset \bigcup_{n\in A_j^m} B(K_n^m, w_m) \quad \text{and} \quad \bigcup_{n\in A_j^m} K_n^m \subset B\big(N(f, b_j^{m,3}), w_m\big).$$

Therefore Proposition 4.5.1 (2) shows that

$$N(f, a_j) \subset N(f, a_j^{m,4}) \subset \bigcup_{n\in A_j^m} B(K_n^m, w_m) \subset \bigcup_{n\in A_j^m} B(K_n, 3w_m),$$

$$\begin{aligned}\bigcup_{n\in A_j^m} K_n \subset \bigcup_{n\in A_j^m} \overline{B}(K_n^m, 2w_m) &\subset B\big(N(f, b_j^{m,3}), 3w_m\big)\\ &\subset B\big(N(f, b_j), 3w_m\big).\end{aligned}$$

∎

Proposition 4.5.3. *If $f \in \bigcap_{m=1}^{\infty} B(g_m, \beta_m)$, then $(\boldsymbol{K}, f) \in \mathscr{X}$.*

Proof. Remember that if $2 \le j \le m-1$, then

$$\bigcup_{n\in A_j^m\setminus A_j^{m-1}} \tilde{K}_n^m \subset \bigcup_{n\in A_{j-1}^{m-1}} B(\tilde{K}_n^m, 2w_{m-1}),$$

and so the same inclusion holds when $\tilde{K}_n^m$ is replaced by K_n^m:

$$\bigcup_{n\in A_j^m\setminus A_j^{m-1}} K_n^m \subset \bigcup_{n\in A_{j-1}^{m-1}} B(K_n^m, 2w_{m-1}).$$

Therefore Proposition 4.5.1 (2) shows that

$$\begin{aligned}\bigcup_{n\in A_j^m\setminus A_j^{m-1}} K_n \subset \bigcup_{n\in A_j^m\setminus A_j^{m-1}} \overline{B}(K_n^m, 2w_m) &\subset \bigcup_{n\in A_{j-1}^{m-1}} B(K_n^m, 2w_m + 2w_{m-1})\\ &\subset \bigcup_{n\in A_{j-1}^{m-1}} B(K_n, 4w_m + 2w_{m-1})\end{aligned}$$

whenever $2 \le j \le m-1$. Hence if we define $\boldsymbol{\delta} \in Y$ by $\delta_m = 4w_m + 2w_{m-1}$ for $m \in \mathbb{N}$, then, using Proposition 4.5.2, we may conclude that

- $\bigcup_{n\in A_j^m\setminus A_j^{m-1}} K_n \subset \bigcup_{n\in A_{j-1}^{m-1}} B(K_n, \delta_m)$ whenever $2 \le j \le m-1$, i.e. $\boldsymbol{K} \in \mathscr{S}_0(\boldsymbol{n}, \boldsymbol{\delta})$;

- $N(f, a_j) \subset \bigcup_{n \in A_j^m} B(K_n, \delta_m)$ whenever $j \leq m$;
- $\bigcup_{n \in A_j^m} K_n \subset B\big(N(f, b_j), \delta_m\big)$ whenever $j \leq m$.

It follows that $(\boldsymbol{K}, f, \boldsymbol{n}, \boldsymbol{\delta}, \boldsymbol{a}, \boldsymbol{b}) \in \mathscr{Y}_0$, implying that $(\boldsymbol{K}, f) \in \mathscr{X}$. ∎

Proposition 4.5.4. *We have* $\bigcap_{m=1}^{\infty} B(g_m, \beta_m) \subset S$. *Hence the strategy makes Player II win.*

Proof. Immediate from Proposition 4.5.1 (3) and Proposition 4.5.3. ∎

This completes the proof of the key proposition (Proposition 4.4.21) and hence the main theorem has been proved.

4.6 Outline of the proof

This section will explain the outline of the proof of the main theorem. Taking the complexity of the strategy for the Banach-Mazur game into consideration, the author believes that this section helps the reader to understand the proof better, though it is technically not necessary for a mathematically complete proof.

4.6.1 What we shall ignore here

It is not difficult to see that neither Proposition 4.3.8 nor Lemma 4.3.9 (nor Proposition 4.3.11, which followed from the two above) holds if $a = b$. However, in the strategy for the Banach-Mazur game, we used them for a and b that are very close to each other, and assuming that they are true even when a equals b helps us a lot to understand the outline of the proof. For this reason, we shall make this assumption in this section, thereby ignoring the difference between $a_j^{m,k}$ and a_j.

Note that Proposition 4.3.8 for $a = b$ shows that the map $C(I) \longrightarrow \mathcal{K}$; $f \longmapsto \tilde{N}(f, a)$ is continuous, whereas Lemma 4.3.9 shows that $\tilde{N}(f, a)$ is open if $f \in C^1(I)$.

4.6.2 Why we need the density condition and the disjoint condition

The first observation is that if we can establish

- $\tilde{N}(g_m, a_j) \subset \bigcup_{n \in A_j} B(\tilde{K}_n^m, w_m)$ for $j \in [m]$;
- $\bigcup_{n \in A_j} \tilde{K}_n^m \subset \tilde{N}(g_m, b_j)$ for $j \in [m]$

in the mth round, where A_j is a finite subset of $\mathbb{N}$ not depending on m, then, assuming $g_m \to g$ and $\tilde{K}_n^m \to \tilde{K}_n$ as $m \to \infty$ and looking at the limits of these relations, we obtain

- $\tilde{N}(g, a_j) \subset \bigcup_{n \in A_j} \tilde{K}_n$ for $j \in \mathbb{N}$;
- $\bigcup_{n \in A_j} \tilde{K}_n \subset \tilde{N}(g, b_j)$ for $j \in \mathbb{N}$.

By taking unions for $j \in \mathbb{N}$ and for $\hat{}$ and $\check{}$, we deduce that $N(g) \subset \bigcup_{n=1}^{\infty} K_n \subset N(g)$, i.e. $N(g) = \bigcup_{n=1}^{\infty} K_n$, assuming that $\bigcup_{j=1}^{\infty} A_j = \mathbb{N}$ and that $K_n = \hat{K}_n \cup \check{K}_n$. If $\boldsymbol{K} \in \bigcap_{n=1}^{\infty} \mathscr{U}_n$, then the strategy makes Player II win.

So let us try to establish

- $\tilde{N}(g_m, a_j) \subset \bigcup_{n \in A_j} B(\tilde{K}_n^m, w_m)$ for $j \in [m]$;
- $\bigcup_{n \in A_j} \tilde{K}_n^m \subset \tilde{N}(g_m, b_j)$ for $j \in [m]$

inductively. Suppose that they are true for $m-1$. By making β_{m-1} small enough, we may assume that f_m satisfies the same relations as g_{m-1}, i.e.

- $\tilde{N}(f_m, a_j) \subset \bigcup_{n \in A_j} B(\tilde{K}_n^{m-1}, w_{m-1})$ for $j \in [m-1]$;
- $\bigcup_{n \in A_j} \tilde{K}_n^{m-1} \subset \tilde{N}(f_m, b_j)$ for $j \in [m-1]$

because of Proposition 4.3.8. In order to establish $\boldsymbol{K} \in \bigcap_{n=1}^{\infty} \mathscr{U}_n$, we may need to move $\tilde{K}_n^{m-1}$ to get $\tilde{K}_n^m$ satisfying $\boldsymbol{K}^m \in \mathscr{U}_m$; however, since $\mathscr{U}_m$ is open dense, we may assume that $\tilde{K}_n^{m-1}$ and $\tilde{K}_n^m$ are close enough to satisfy the same relations, i.e.

- $\tilde{N}(f_m, a_j) \subset \bigcup_{n \in A_j} B(\tilde{K}_n^m, w_{m-1})$ for $j \in [m-1]$;

- $\bigcup_{n\in A_j} \tilde{K}_n^m \subset \tilde{N}(f_m, b_j)$ for $j \in [m-1]$.

We are going to construct g_m by adding to f_m a bump function φ_m located at some sets $\hat{H}$ and $\check{H}$. Since Proposition 4.3.16 shows that $\tilde{H} \cap \tilde{N}(f_m, b_j) \subset \tilde{N}(g_m, b_j)$, we can obtain the second condition that we wish to establish, provided that $\bigcup_{n\in A_j} \tilde{K}_n^m \subset \tilde{H}$ for $j \in [m-1]$ (here we neglect the case $j = m$, which is not too difficult to deal with). Proposition 4.3.17 helps us to obtain the first condition, but there are two obstacles to overcome. Firstly, in order for the proposition to be applicable, the location $\tilde{H}$ of the bump function φ_m must satisfy the *density condition*

- $B(\tilde{H}, \mu_m) = I$,

where μ_m depends on the function f_m, the numbers a_j, and the height of φ_m only. Secondly, in order to obtain the desired relation $\tilde{N}(g_m, a_j) \subset \bigcup_{n\in A_j} B(\tilde{K}_n^m, w_m)$ after getting the relation $\tilde{N}(g_m, a_j) \subset \tilde{N}(f_m, a_j) \cap B(\tilde{H}, w_m)$ that the proposition implies (w_m is the width of φ_m), the *disjoint condition*

- $\tilde{N}(f_m, a_j) \cap \bigcup_{n\in A_{m-1}\setminus A_j} B(\tilde{K}_n^m, w_m) = \emptyset$

must hold because, for the aforementioned reason, the location $\tilde{H}$ must be large enough to contain $\bigcup_{n\in A_j} \tilde{K}_n^m$ for all $j \in [m-1]$.

4.6.3 Why we need A_j^m rather than A_j

It turns out that the second obstacle in itself does not worry us very much. Let us strengthen what we prove by induction, and try to prove

- $\tilde{N}(g_m, a_j) \subset \bigcup_{n\in A_j} B(\tilde{K}_n^m, w_m)$ for $j \in [m]$;
- $\bigcup_{n\in A_j} \tilde{K}_n^m \subset \tilde{N}(g_m, b_j)$ for $j \in [m]$;
- $\tilde{N}(g_m, a_j) \cap \bigcup_{n\in A_m\setminus A_j} \overline{B}(\tilde{K}_n^m, w_m) = \emptyset$ for $j \in [m]$

(we changed B into $\overline{B}$ in the disjoint condition; though not essential, it makes our life slightly easier). Assuming that they are true for $m-1$, we may prove the first and the second conditions for m as described above, if we set $\tilde{H} = \bigcup_{n \in A_{m-1}} \tilde{K}_n^m$. The disjoint condition follows from the first condition if we make w_m so small that the balls $\overline{B}(x, w_m)$ for $x \in \bigcup_{n \in A_m} \tilde{K}_n^m$ are disjoint, because we always choose finite sets as $\tilde{K}_n^m$.

However, when we try to overcome the first obstacle as well, we face a great trouble. Since $\bigcup_{n \in A_{m-1}} \tilde{K}_n^m$ may not be large enough to satisfy the density condition $B(\bigcup_{n \in A_{m-1}} \tilde{K}_n^m, \mu_m) = I$, we have to add points to make $\tilde{H}$. Denote by $\tilde{M}$ the set of added points: $\tilde{H} = \bigcup_{n \in A_{m-1}} \tilde{K}_n^m \cup \tilde{M}$. In order for $\tilde{H}$ to satisfy the density condition $B(\tilde{H}, \mu_m) = I$, the set $\tilde{M}$ must satisfy

$$B(\tilde{M}, \mu_m) \supset I \setminus \bigcup_{n \in A_{m-1}} B(\tilde{K}_n^m, \mu_m).$$

On the other hand, for the same reason as the necessity of the disjoint condition, the set $\tilde{M}$ must satisfy $\tilde{N}(f_m, a_j) \cap B(\tilde{M}, w_m) = \emptyset$ for $j \in [m-1]$. Since w_m will be defined later and $\tilde{N}(f_m, a_j)$ increases as j does, the condition that we should impose on $\tilde{M}$ is

$$\tilde{N}(f_m, a_{m-1}) \cap \tilde{M} = \emptyset.$$

Can we choose $\tilde{M}$ satisfying these two conditions? Unfortunately, the answer is negative for the following reason. Since μ_m depends on f_m, which was chosen after w_{m-1} was defined, it may be that μ_m is much smaller than w_{m-1}. Keeping in mind that $\tilde{N}(f_m, a_{m-1}) \subset \bigcup_{n \in A_{m-1}} B(\tilde{K}_n^m, w_{m-1})$ is the only relation that we currently know, we must be prepared for the case where $\bigcup_{n \in A_{m-1}} B(\tilde{K}_n^m, \mu_m)$ fails to cover $\tilde{N}(f_m, a_{m-1})$. It means that if we set $S = \tilde{N}(f_m, a_{m-1}) \setminus \bigcup_{n \in A_{m-1}} B(\tilde{K}_n^m, \mu_m)$, then S may be large; S might contain an open interval of length $2\mu_m$, in which case there is no $\tilde{M}$ for which both $S \cap \tilde{M} = \emptyset$ and $B(\tilde{M}, \mu_m) \supset S$ are true. It follows that we may not be able to choose $\tilde{M}$ with the desired properties.

In order to sort this problem out, we try to find the sets $\tilde{K}_n^m$ that approximate $\tilde{N}(f_m, a_j)$ better. That is to say, using the relations

- $\tilde{N}(f_m, a_j) \subset \bigcup_{n \in A_j} B(\tilde{K}_n^{m-1}, w_{m-1})$ for $j \in [m-1]$;

- $\bigcup_{n \in A_j} \tilde{K}_n^{m-1} \subset \tilde{N}(f_m, b_j)$ for $j \in [m-1]$

that we currently know, we try to find $\tilde{K}_n^m$ such that

- $\tilde{N}(f_m, a_j) \subset \bigcup_{n \in A_j} B(\tilde{K}_n^m, \mu_m)$ for $j \in [m-1]$;
- $\bigcup_{n \in A_j} \tilde{K}_n^m \subset \tilde{N}(f_m, b_j)$ for $j \in [m-1]$.

It might seem to be a good idea to define $\tilde{K}_n^m = \tilde{N}(f_m, a_j) \cap B(\tilde{K}_n^{m-1}, w_{m-1})$ if $n \in A_j$ (the right-hand side is not closed, but this can be remedied easily) because it implies that

$$\tilde{N}(f_m, a_j) = \bigcup_{n \in A_j} \big(\tilde{N}(f_m, a_j) \cap B(\tilde{K}_n^{m-1}, w_{m-1})\big) = \bigcup_{n \in A_j} \tilde{K}_n^m,$$

$$\bigcup_{n \in A_j} \tilde{K}_n^m \subset \tilde{N}(f_m, a_j) \subset \tilde{N}(f_m, b_j)$$

(although we might seem to have established much better approximation because no μ_m appears here, we are now ignoring the requirement that $\boldsymbol{K}^m$ should belong to $\mathscr{U}_m$ and it forces us to change $\tilde{K}_n^m$ slightly, which yields μ_m). What we missed here is the fact that each n might belong to more than one A_j; we must decide which j we should use to define $\tilde{K}_n^m$. For example, suppose that an integer n belongs to both A_1 and A_2. On the one hand, if we define $\tilde{K}_n^m = \tilde{N}(f_m, a_1) \cap B(\tilde{K}_n^{m-1}, w_{m-1})$, then $\tilde{K}_n^m$ can be too small for $\bigcup_{n' \in A_2} B(\tilde{K}_{n'}^m, \mu_m)$ to cover $\tilde{N}(f_m, a_2)$; on the other hand, if we define $\tilde{K}_n^m = \tilde{N}(f_m, a_2) \cap B(\tilde{K}_n^{m-1}, w_{m-1})$, then $\tilde{K}_n^m$ can be too large for $\bigcup_{n' \in A_1} \tilde{K}_{n'}^m$ to be contained in $\tilde{N}(f_m, b_1)$.

What we do to solve this problem is to define $\tilde{K}_n^m$ as above with the minimum j with $n \in A_j$, and introduce $\tilde{K}_n^m$ for n larger than $\max A_{m-1}$ that will be in charge of those *bad* points that belong to $\tilde{N}(f_m, a_j) \setminus \tilde{N}(f_m, a_{j-1})$ but whose nearest point in $\bigcup_{n \in A_j} \tilde{K}_n^{m-1}$ belongs to $\bigcup_{n \in A_{j-1}} \tilde{K}_n^{m-1}$, not $\bigcup_{n \in A_j \setminus A_{j-1}} \tilde{K}_n^{m-1}$. According to this construction, the exact relation $\tilde{N}(f_m, a_j) \subset \bigcup_{n \in A_j} B(\tilde{K}_n^m, \mu_m)$ cannot be established, but if we add the large numbers n to A_j on the right-hand side, then the inclusion becomes true; we must use A_j^m rather than A_j to show the dependence on m as well as j.

4.6.4 What we should be careful about when using A_j^m

In the way described above, we can establish the following:

- $\tilde{N}(g_m, a_j) \subset \bigcup_{n\in A_j^m} B(\tilde{K}_n^m, w_m)$ for $j \in [m]$;
- $\bigcup_{n\in A_j^m} \tilde{K}_n^m \subset \tilde{N}(g_m, b_j)$ for $j \in [m]$;
- $\tilde{N}(g_m, a_j) \cap \bigcup_{n\in A_m^m \setminus A_j^m} \overline{B}(\tilde{K}_n^m, w_m) = \emptyset$ for $j \in [m]$.

Taking unions for $\hat{}$ and $\check{}$ in the first two relations gives

- $N(g_m, a_j) \subset \bigcup_{n\in A_j^m} B(K_n^m, w_m)$ for $j \in [m]$;
- $\bigcup_{n\in A_j^m} K_n^m \subset N(g_m, b_j)$ for $j \in [m]$.

We attempt to use these relations to prove that $\bigcup_{n=1}^\infty K_n = N(g)$, where $g = \lim_{m\to\infty} g_m$, whose existence we may assume without loss of generality. It is natural to assume that $\bigcup_{j=1}^\infty \bigcup_{m=j}^\infty A_j^m = \mathbb{N}$ and that $w_m \to 0$ as $m \to \infty$, if we consider their roles in the strategy. Note that A_j^m increases as m does because of the construction.

The inclusion $\bigcup_{n=1}^\infty K_n \subset N(g)$ can be proved easily in the following manner. Let $x \in \bigcup_{n=1}^\infty K_n$ and take $n \in \mathbb{N}$ with $x \in K_n$. Since $K_n^m \to K_n$ as $m \to \infty$, we can find points $x_m \in K_n^m$ that converge to x as $m \to \infty$. By the assumption $\bigcup_{j=1}^\infty \bigcup_{m=j}^\infty A_j^m = \mathbb{N}$, we may take $j \in \mathbb{N}$ with $n \in \bigcup_{m=j}^\infty A_j^m$. Then, for sufficiently large m, we have $n \in A_j^m$ and so

$$x_m \in K_n^m \subset \bigcup_{n'\in A_j^m} K_{n'}^m \subset N(g_m, b_j).$$

Since $N(g_m, b_j) \to N(g, b_j)$ as $m \to \infty$, it follows that

$$x = \lim_{m\to\infty} x_m \in N(g, b_j) \subset N(g).$$

However, the opposite inclusion $N(g) \subset \bigcup_{n=1}^\infty K_n$ is not so easy to prove. Let $x \in N(g) = \bigcup_{j=1}^\infty N(g, a_j)$ and take $j \in \mathbb{N}$ with $x \in N(g, a_j)$. Since $N(g_m, a_j) \to N(g, a_j)$ as $m \to \infty$, we can find points $x_m \in N(g_m, a_j)$ that converge to x as $m \to \infty$. For each $m \geq j$, because $x_m \in N(g_m, a_j) \subset \bigcup_{n\in A_j^m} B(K_n^m, w_m)$, there exists $n_m \in A_j^m$ with $x_m \in B(K_{n_m}^m, w_m)$; take $y_m \in K_{n_m}^m$ with $|x_m - y_m| < w_m$. Since $x_m \to x$ and $w_m \to 0$, we have $y_m \to x$. If there exists $n \in \mathbb{N}$ such that $n_m = n$ for infinitely many m, then $x \in K_n$ because $y_m \in K_n^m$ for m with $n_m = n$, and we are done. However, if such n does not exist, i.e. if $n_m \to \infty$ as $m \to \infty$, then it may be that no K_n contains x.

Fortunately, we can solve this problem by looking at the properties of the sets K_n^m more closely without changing their construction. Suppose that $n_m \to \infty$ as $m \to \infty$. Since $n_m \in A_j^m$ for all m, we can, for sufficiently large m, consider n_m to be added indices introduced to take care of *bad* points. Therefore $y_m \in K_{n_m}^m$ must be close to some point in $\bigcup_{n \in A_{j-1}^{m-1}} K_n^{m-1}$, from which we may infer that x does belong to some K_n; see Proposition 4.4.10 for details.

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