Continuity Points of Typical Functions

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Abstract

Since Banach and Mazurkiewicz independently proved that typical (in the sense of Baire category) continuous functions are nowhere differentiable, the study of the behaviour of typical continuous functions has been one of the most popular topics in classical real analysis. Despite being less popular, it is also interesting and important to investigate typical members of other families of functions. The talk will look at several families in which typical functions have small continuity points.

1 Continuity points of functions of Baire class 1

By a *function* we shall always mean a function from the real line $\mathbb{R}$ into itself.

All mathematics students are taught in their first year that a pointwise limit of continuous functions is not necessarily continuous, as illustrated by the following example.

Example 1.1.
Define a sequence $\{f_n\}$ of continuous functions by

$$f_n(x) = \begin{cases} 0 & \text{if } |x| > 1/n; \\ 1 - n|x| & \text{if } |x| \leq 1/n. \end{cases}$$

Then $\{f_n\}$ converges pointwise to the characteristic function of $\{0\}$, which is discontinuous at 0.

Definition 1.2.
A function is said to be of **Baire class 1** if it can be expressed as a pointwise limit of continuous functions. The family of all functions of Baire class 1 will be denoted by B_1.

All continuous functions are of Baire class 1; the converse fails as we saw in Example 1.1.

Definition 1.3.
For each function f, we write

$$C(f) = \{x \in \mathbb{R} \mid f \text{ is continuous at } x\},$$
$$D(f) = \{x \in \mathbb{R} \mid f \text{ is not continuous at } x\}.$$

The function $f \in B_1$ in Example 1.1 satisfies $D(f) = \{0\}$. We may well ask ourselves whether a function $f \in B_1$ can have much bigger $D(f)$; for example, is there $f \in B_1$ with $D(f) = \mathbb{R}$? It turns out that $D(f)$ is topologically small for every $f \in B_1$. In order to state this proposition precisely, we need a few terms.

Definition 1.4.
Let X be a topological space and A a subset of X.

(1) We say that A is **nowhere dense** if $\operatorname{Int} \bar{A} = \emptyset$.

(2) We say that A is **meagre** if A can be written as a countable union of nowhere dense sets.

Proposition 1.5 (Baire category theorem).
In a complete metric space, every meagre set has empty interior.

This proposition means that we can regard meagreness as the mathematically rigorous concept for topological smallness in complete metric spaces. Note that it is not the case for all topological spaces; for example, the whole space is meagre in $\mathbb{Q}$, where meagreness makes no sense.

Proposition 1.6.
If $f \in B_1$, then $D(f)$ is meagre.

Proof.
Define the **oscillation** $\operatorname{osc}(f, a)$ of f at a point $a \in \mathbb{R}$ by

$$\operatorname{osc}(f, a) = \inf_{\delta > 0} \sup_{x, y \in (a-\delta, a+\delta)} |f(x) - f(y)| \in [0, \infty],$$

and set $A_\varepsilon = \{x \in \mathbb{R} \mid \operatorname{osc}(f, x) \geq \varepsilon\}$ for each $\varepsilon > 0$. It is easy to see that each A_ε is closed and that $D(f) = \{x \in \mathbb{R} \mid \operatorname{osc}(f, x) > 0\} = \bigcup_{m=1}^{\infty} A_{1/m}$. Therefore it suffices to prove that $\operatorname{Int} A_\varepsilon = \emptyset$ for all $\varepsilon > 0$. Suppose for a contradiction that some A_ε contains a nondegenerate closed interval I.

Take a sequence $\{f_n\}$ of continuous functions converging pointwise to f, and look at the closed set

$$B_n = \bigcap_{i,j=n}^{\infty} \bigl\{x \in \mathbb{R} \bigm| |f_i(x) - f_j(x)| \le \varepsilon/4\bigr\}$$

for each $n \in \mathbb{N}$. Since $I = \bigcup_{n=1}^{\infty}(B_n \cap I)$ by the pointwise convergence of $\{f_n\}$, the Baire category theorem implies that $B_n \cap I$ contains a nondegenerate closed interval, say J, for some $n \in \mathbb{N}$. For each $x \in J$, we have $|f_i(x) - f_n(x)| \le \varepsilon/4$ for all $i \ge n$, and so $|f(x) - f_n(x)| \le \varepsilon/4$. The uniform continuity of f_n on J allows us to take $\delta > 0$ less than half the length of J so that $|f_n(x) - f_n(y)| \le \varepsilon/4$ whenever $x, y \in J$ and $|x - y| < 2\delta$.

Now let a be the midpoint of J. If $x, y \in (a - \delta, a + \delta) \subset J$, then

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \le 3\varepsilon/4.$$

It follows that $\operatorname{osc}(f, a) \le 3\varepsilon/4 < \varepsilon$, contradicting the fact that $a \in J \subset I \subset A_\varepsilon$. ∎

2 Continuity points of typical functions of Baire class 1

Having proved in Section 1 that $D(f)$ is topologically small for every $f \in B_1$, we are tempted to know whether $D(f)$ is also measure-theoretically small, i.e. Lebesgue null, for every $f \in B_1$. Contrary to the intuition one may have, Bruckner and Petruska [BP] proved that $C(f)$, rather than $D(f)$, is Lebesgue null for *most* $f \in B_1$.

In order to define what we mean by *most* $f \in B_1$, we need to give B_1 a topology. For ease of presentation, we restrict ourselves to *bounded* functions of Baire class 1 and consider

$$bB_1 = \{f \in B_1 \mid f \text{ is bounded}\},$$

equipped with the supremum norm $\|f\| = \sup_{x\in\mathbb{R}}|f(x)|$.

The following proposition assures us that meagreness makes sense in bB_1:

Proposition 2.1.
The space bB_1 is a Banach space.

Proof.

It suffices to show that if $f_n \in B_1$ and $f_n \to f$ uniformly, then $f \in B_1$.

By taking a subsequence if necessary, we may assume that $\|f_n - f\| < 2^{-n}$ for all $n \in \mathbb{N}$. Put $g_n = f_{n+1} - f_n \in B_1$ for each $n \in \mathbb{N}$, and $g = f - f_1$. We have $g = \sum_{n=1}^{\infty} g_n$ uniformly, and it suffices to show that $g \in B_1$.

Take continuous functions $\{g_{mn}\}_{m,n\in\mathbb{N}}$ so that $g_{mn} \to g_n$ pointwise as $m \to \infty$ for each $n \in \mathbb{N}$. Since

$$\|g_n\| \le \|f_{n+1} - f\| + \|f - f_n\| < 2^{-n-1} + 2^{-n} < 2^{-n+1},$$

we may assume that $\|g_{mn}\| \leq 2^{-n+1}$ for all $m, n \in \mathbb{N}$ by replacing g_{mn} with

$$\max\left\{\min\{g_{mn}, 2^{-n+1}\}, -2^{-n+1}\right\}$$

if necessary.

Set $h_m = \sum_{n=1}^{m} g_{mn}$ for each $m \in \mathbb{N}$. We shall show that $h_m \to g$ pointwise, which implies that $g \in B_1$ as required. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be given. Choose $M \in \mathbb{N}$ so large that $\sum_{n=M+1}^{\infty} 2^{-n+1} < \varepsilon/3$, and then choose $M' > M$ so large that $|g_{mn}(x) - g_n(x)| \leq \varepsilon/3M$ for $n = 1, \ldots, M$ whenever $m \geq M'$. For $m \geq M'$, we have

$$\begin{aligned}
|h_m(x) - g(x)| &= \left|\sum_{n=1}^{m} g_{mn}(x) - \sum_{n=1}^{\infty} g_n(x)\right| \\
&\leq \sum_{n=1}^{M} |g_{mn}(x) - g_n(x)| + \sum_{n=M+1}^{m} |g_{mn}(x)| + \sum_{n=M+1}^{\infty} |g_n(x)| \\
&\leq M \cdot \frac{\varepsilon}{3M} + \sum_{n=M+1}^{m} 2^{-n+1} + \sum_{n=M+1}^{\infty} 2^{-n+1} \\
&< \varepsilon,
\end{aligned}$$

completing the proof. ∎

Definition 2.2.
We say that a **typical** $f \in bB_1$ has a property P, written $\forall^* f \in bB_1\ P$, if

$$\{f \in bB_1 \mid f \text{ does not have property } P\}$$

is meagre in bB_1.

The following is the theorem of Bruckner and Petruska mentioned at the beginning of the present section:

Theorem 2.3 ([BP, Theorem 2.4], weakened for simplicity).
A typical $f \in bB_1$ has the property that $C(f)$ is Lebesgue null.

3 Theorem of Kostyrko and Šalát, and our main theorem

We now turn our attention from bB_1 to general spaces of bounded functions. Let X be a linear space of bounded functions, equipped with the supremum norm.

Definition 3.1.
We say that a **typical** $f \in X$ has a property P, written $\forall^* f \in X\ P$, if

$$\{f \in X \mid f \text{ does not have property } P\}$$

is meagre in X.

Note that even when a typical $f \in X$ has a property P, there does not necessarily exist $f \in X$ with the property P, since we do not assume X to be closed under uniform convergence.

Kostyrko and Šalát [KS] investigated in which families X it is true that a typical $f \in X$ has the property that $C(f)$ is Lebesgue null.

Theorem 3.2 ([KS, Theorem], weakened for simplicity).
If $C(f)$ is Lebesgue null for some $f \in X$, then a typical $f \in X$ has the property that $C(f)$ is Lebesgue null.

Our main theorem shows that the theorem above remains valid when 'Lebesgue null' is replaced by 'meagre.'

Theorem 3.3 (Main Theorem, [Sa, Theorem 1.3], weakened for simplicity).
If $C(f)$ is meagre for some $f \in X$, then a typical $f \in X$ has the property that $C(f)$ is meagre.

References

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