正規コピュラの漸近的裾依存性

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1. コピュラとは

確率変数間の依存関係を記述

Def 1.1

A **copula** is $C : [0,1]^2 \to [0,1]$

s.t.

$\exists U, V \sim \mathrm{Uniform}[0,1]$

(not necessarily independent)

$$C(u, v) = P(U \leqq u, V \leqq v)$$

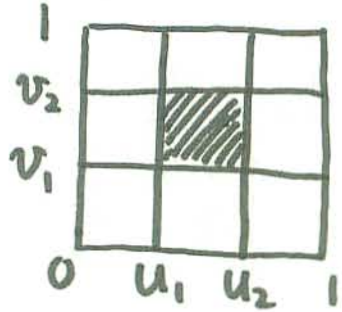
$$(\forall u, v \in [0,1])$$

or equivalently

$$\begin{cases} \bullet\ \forall u, v \in [0,1] \\ \quad C(u, 0) = C(0, v) = 0 \\ \quad C(u, 1) = u,\ C(1, v) = v \end{cases}$$

- $0 \leqq u_1 \leqq u_2 \leqq 1,\ 0 \leqq v_1 \leqq v_2 \leqq 1$

$$\Rightarrow C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geqq 0$$



X, Y : $\mathbb{R}$-valued r.v.'s

$F_{X,Y}(x, y) := P(X \leqq x, Y \leqq y)$

$F_X(x) := P(X \leqq x)$

$F_Y(y) := P(Y \leqq y)$

Assume F_X, F_Y : conti

Then

$F_X(X), F_Y(Y) \sim \text{Uniform}[0, 1]$

Thm 1.3 (Sklar)

$\exists! C$: copula

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)) \quad (\forall x, y \in \mathbb{R})$$

[Denote this C by $C_{X,Y}$.]

(Prf)

$C(u, v) := P(F_X(X) \leqq u, F_Y(Y) \leqq v)$: copula.

Then

$$\begin{aligned} F_{X,Y}(x, y) &= P(X \leqq x, Y \leqq y) \\ &= P(F_X(X) \leqq F_X(x), F_Y(Y) \leqq F_Y(y)) \\ &= C(F_X(x), F_Y(y)) \end{aligned}$$

$\blacksquare$

e.g. 1.4

X, Y: independent

$\iff C_{X,Y}(u,v) = uv$ product copula

Def 1.5

Let $-1 < \rho < 1$.

$(X,Y) \sim \text{Normal}\left((0,0), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$

$\Rightarrow C_{X,Y}$: **normal copula** with correlation ρ

$\quad\; \overset{\shortparallel}{C_\rho}$.

2. コピュラの裾依存性

X, Y: $\mathbb{R}$-valued r.v.'s

Assume F_X, F_Y : conti.

Then

$F_X(X), F_Y(Y) \sim \text{Uniform}[0,1]$

Def 2.1

For $0 < t < 1$,

$$\lambda_{X,Y}(t) := P\left(F_Y(Y) > t \,\middle|\, F_X(X) > t\right)$$

$$\lambda_{X,Y} := \lim_{t \uparrow 1} \lambda_{X,Y}(t)$$

e.g.

- X, Y : independent

  $\Rightarrow \lambda_{X,Y}(t) = 1 - t \rightarrow \lambda_{X,Y} = 0$

- $X = Y$

  $\Rightarrow \lambda_{X,Y}(t) = 1 \rightarrow \lambda_{X,Y} = 1$

Prop 2.2

$$\lambda_{X,Y}(t) = \frac{1 - 2t + C_{X,Y}(t,t)}{1-t}$$

$\therefore \lambda_{X,Y}(t)$ and $\lambda_{X,Y}$ depend only on $C_{X,Y}$.

e.g. (6)

$C_{X,Y} = C_\rho$ (normal copula)

$\Rightarrow \lambda_{X,Y} = 0$

"No tail dependence?"

t	$\lambda_{X,Y}(t)$ prod cop	$C_\rho(\rho=0.5)$
0.8	0.2000	0.4358
0.9	0.1000	0.3240
0.95	0.0500	0.2438
0.99	0.0100	0.1294
0.995	0.0050	0.0993
0.999	0.0010	0.0543

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Different asymptotic tail dependence!

Main Thm (7)

$C_{X,Y} = C_\rho$

$$\Rightarrow \lambda_{X,Y}(t) = \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, e^{-\frac{1-\rho}{2(1+\rho)}s^2}\left(\frac{1}{s} - \frac{1+2\rho-\rho^2}{1-\rho}\cdot\frac{1}{s^3} + O\left(\frac{1}{s^5}\right)\right)$$

as $t \uparrow 1$

$$\left[\begin{array}{l} s = \Phi^{-1}(t) \uparrow \infty \\ t = \Phi(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} e^{-\frac{x^2}{2}}\,dx \end{array}\right]$$

$$\sim \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, e^{-\frac{1-\rho}{2(1+\rho)}s^2}\,\frac{1}{s}$$

$$\sim (2\pi)^{-\frac{\rho}{1+\rho}}\sqrt{\frac{(1+\rho)^3}{1-\rho}}\,(1-t)^{\frac{1-\rho}{1+\rho}}\left(\Phi^{-1}(t)\right)^{-\frac{2\rho}{1+\rho}}$$

$$\left(\because\ 1-t \sim \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{s}\cdot e^{-\frac{s^2}{2}}\right)$$

$$\sim (4\pi)^{-\frac{\rho}{1+\rho}}\sqrt{\frac{(1+\rho)^3}{1-\rho}}\,(1-t)^{\frac{1-\rho}{1+\rho}}\left(-\log(1-t)\right)^{-\frac{\rho}{1+\rho}}$$

$$\left(\because\ s = \Phi^{-1}(t) \sim \left(-2\log(1-t)\right)^{\frac{1}{2}}\right)$$