Cauchy Functional Equation

1 Introduction

In this note, we shall prove that if $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the *Cauchy functional equation*

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$ and if f is Lebesgue measurable, then $f(x) = xf(1)$ for all $x \in \mathbb{R}$. We may easily observe that the Cauchy functional equation is equivalent to $\mathbb{Q}$-linearity, and so we know $f(x) = xf(1)$ for all $x \in \mathbb{Q}$ even if f may not be Lebesgue measurable.

The first volume of *Fundamenta Mathematicae* contains two papers, each written by Sierpiński and by Banach, that give completely different proofs of this theorem.

In what follows, the Lebesgue measure will be denoted by μ.

2 Proof by Banach

Theorem 2.1. *If $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Cauchy functional equation and is Lebesgue measurable, then $f(x) = xf(1)$ for all $x \in \mathbb{R}$.*

Proof. Notice first that it suffices to prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(h)| < \varepsilon$ for all $h \in (0, \delta)$. This is because it implies the continuity of f at 0, from which we may infer the continuity of f.

Let $\varepsilon > 0$ be given. Since f is Lebesgue measurable, Lusin's theorem allows us to find a closed set $F \subset [0, 1]$ with $\mu(F) \geqq 2/3$ on which f is continuous. As f is uniformly continuous on F, there exists $\delta \in (0, 1/3)$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in F$ and $|x - y| < \delta$. Let $h \in (0, \delta)$. If F and $F - h = \{\, x - h \mid x \in F \,\}$ were disjoint, then we would have

$$1 + h = \mu([-h, 1]) \geqq \mu\bigl(F \cup (F - h)\bigr) = \mu(F) + \mu(F - h) \geqq 4/3,$$

which contradicts $h < \delta < 1/3$. So we may take a point $x_0 \in F \cap (F - h)$, and we see that $|f(x_0) - f(x_0 + h)| < \varepsilon$, that is, $|f(h)| < \varepsilon$. ∎

3 Proof by Sierpiński

Lemma 3.1. *If $A, B \subset \mathbb{R}$ have positive Lebesgue measure, then there exist $a \in A$ and $b \in B$ such that $a - b \in \mathbb{Q}$.*

Proof. We may assume that A and B are bounded. By an interval we shall mean a compact nondegenerate interval. Take an interval I containing B, and choose $\varepsilon > 0$ less than $\mu(A)\mu(B)/3\mu(I)$.

First we claim that there exists an interval J with rational endpoints of length less than $\mu(I)$ satisfying

$$\mu(A \cap J) > \frac{\mu(A)}{\mu(A) + \varepsilon} \mu(J).$$

Suppose that such J does not exist. Choose intervals J_1, J_2, ... with rational endpoints of length less than $\mu(I)$ so that $\bigcup_{j=1}^{\infty} J_j \supset A$ and $\sum_{j=1}^{\infty} \mu(J_j) < \mu(A) + \varepsilon$. Then, by assumption and $\mu(A) > 0$, we obtain

$$\mu(A) = \mu\left(\bigcup_{j=1}^{\infty} (A \cap J_j)\right) \leqq \sum_{j=1}^{\infty} \mu(A \cap J_j) \leqq \frac{\mu(A)}{\mu(A) + \varepsilon} \sum_{j=1}^{\infty} \mu(J_j) < \mu(A),$$

a desired contradiction. Thus the claim has been proved.

Let S denote the set of all integers n for which $J + n\mu(J)$ meets I. We put $s = \#S$, keeping in mind that S is finite. Then we have $(s-2)\mu(J) \leqq \mu(I)$, and so $s\mu(J) \leqq \mu(I) + 2\mu(J) < 3\mu(I)$. If B and $\bigcup_{n \in S} \big((A \cap J) + n\mu(J)\big)$ were disjoint, then we would have

$$\begin{aligned} s\mu(J) = \mu\left(\bigcup_{n \in S} \big(J + n\mu(J)\big)\right) &\geqq \mu(B) + \mu\left(\bigcup_{n \in S} \big((A \cap J) + n\mu(J)\big)\right) \\ &= \mu(B) + s\mu(A \cap J) > \mu(B) + s\frac{\mu(A)}{\mu(A) + \varepsilon} \mu(J), \end{aligned}$$

which implies $\mu(B) < s\varepsilon\mu(J)/\big(\mu(A) + \varepsilon\big) < 3\varepsilon\mu(I)/\mu(A)$, contradicting the choice of ε. Accordingly we may take a point $b \in B \cap \bigcup_{n \in S} \big((A \cap J) + n\mu(J)\big)$. Since $\mu(J) \in \mathbb{Q}$, the point b is of rational distance from some point $a \in A \cap J$. ∎

Theorem 3.2. *If $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Cauchy functional equation and is Lebesgue measurable, then $f(x) = xf(1)$ for all $x \in \mathbb{R}$.*

Proof. Set $g(x) = f(x) - xf(1)$. Then g is also $\mathbb{Q}$-linear and $g(x) = 0$ for all $x \in \mathbb{Q}$. What we need to show is that $g(x) = 0$ for all $x \in \mathbb{R}$.

Set $A = \{\, x \in \mathbb{R} \mid g(x) > 0 \,\}$ and $B = \{\, x \in \mathbb{R} \mid g(x) < 0 \,\}$. Since $x \longmapsto -x$ is a bijection between A and B, these two sets have the same Lebesgue measure. If $\mu(A) = \mu(B) > 0$, then the lemma would contradict the fact that $g(x) = g(y)$ whenever $x - y \in \mathbb{Q}$. Consequently we have $\mu(A) = \mu(B) = 0$ and so $g^{-1}(0)$ is conull.

If there were a point $a \in \mathbb{R}$ with $g(a) \neq 0$, then the set of all $x \in \mathbb{R}$ with $g(x - a) = 0$ would be a conull set contained in the null set $A \cup B$, which is absurd. Hence such a does not exist. ∎

References

[1] S. Banach, *Sur l'équation fonctionnelle $f(x+y) = f(x)+f(y)$*, Fundamenta Mathematicae, **1** (1920), available at `http://matwbn.icm.edu.pl/ksiazki/fm/fm1/fm1115.pdf`.

[2] W. Sierpiński, *Sur l'équation fonctionnelle $f(x+y) = f(x)+f(y)$*, Fundamenta Mathematicae, **1** (1920), available at `http://matwbn.icm.edu.pl/ksiazki/fm/fm1/fm1114.pdf`.