Asymptotic Tail Dependence of the Normal Copula

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Introduction

- **copula** = something that connects

  [Linguistics]

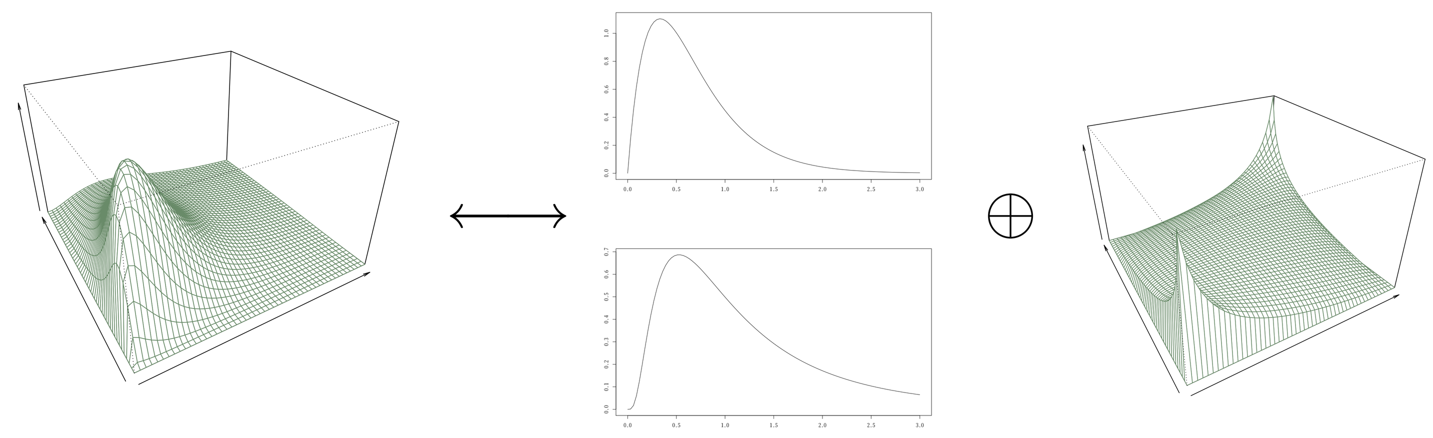
  A copula (= linking verb) connects a **subject** and a **complement**.

  Examples: Hawai'i *is* great. That *sounds* interesting.

  [Mathematics]

  A copula connects a **joint distribution** and its **marginal distributions**.

{n-dim joint distributions} = {marginal distributions}^{n} $\oplus$ {copulas}



- Copulas are – useful for **dependence modeling**;
  – widely used in finance and insurance.

  Examples: risk aggregation and pricing of credit derivatives.

  In such applications,

  tail dependence (= dependence between large values)

  is very important.
- The **normal copula** (a.k.a. **Gaussian copula**) is

☺ constructed from the multivariate normal distribution;

☺ easy to implement;

☹ said to **lack tail dependence**

on the grounds that a certain limit as $t \nearrow 1$ is 0.

- However, what is used in applications is the value at $t = 0.99$, say.

Aim

To look more closely at the tail dependence of the normal copula.

What is a copula?

Definition

A **copula** is $C\colon [0,1]^2 \longrightarrow [0,1]$ s.t.

$\exists$ random variables $U, V \sim \mathrm{Uniform}(0,1)$ (not necessarily independent) s.t.

$$C(u,v) = P(U \le u,\ V \le v) \quad \forall u, v \in [0,1].$$

Sklar's theorem

Sklar's Theorem (1) ← Joint decomposes into marginals and copula

(X,Y): a continuous bivariate random variable.

(Being continuous means that its marginal distribution functions

$$F_X(x) := P(X \le x), \quad F_Y(y) := P(Y \le y)$$

are both continuous.)

$F_{X,Y}(x,y) := P(X \le x,\ Y \le y)$: its joint distribution function.

Then $\exists$**!**$C_{X,Y}$: copula s.t.

$$F_{X,Y}(x,y) = C_{X,Y}(F_X(x), F_Y(y)) \quad \forall x, y \in \mathbb{R}.$$

Sklar's Theorem (2) ← Marginals & copula can be chosen "independently"

F_1, F_2: continuous univariate distribution functions.

C: a copula.

Then $\exists (X,Y)$: bivariate random variable s.t.

- its marginal distributions are F_1 and F_2;
- $C_{X,Y} = C$.

Examples of copulas

- **product copula**: $C(u,v) = uv$ ⟵ corresponds to **independence**.
- **upper Fréchet-Hoeffding bound**: $C(u,v) = \min\{u,v\}$
  ⟵ corresponds to $Y = \varphi(X)$ with φ increasing.
  (**complete positive dependence**).
- **lower Fréchet-Hoeffding bound**: $C(u,v) = \max\{u+v-1, 0\}$
  ⟵ corresponds to $Y = \varphi(X)$ with φ decreasing
  (**complete negative dependence**).
- **normal copula** (**Gaussian copula**): $C_\rho = C_{X,Y}$ for $-1 < \rho < 1$,
  where $(X,Y) \sim N\left((0,0), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$.

Tail dependence and copulas

Let (X,Y) be a continuous bivariate random variable.
Then $F_X(X), F_Y(Y) \sim \mathrm{Uniform}(0,1)$.

Definition

$\lambda_{X,Y}(t) := P(F_Y(Y) > t \mid F_X(X) > t)$ for $0 < t < 1$.

$\lambda_{X,Y} := \lim_{t \nearrow 1} \lambda_{X,Y}(t)$: **tail-dependence parameter**.

Proposition

$\lambda_{X,Y}(t)$ and $\lambda_{X,Y}$ are determined by $C_{X,Y}$ only.

Examples of tail dependence

Write $\lambda_{X,Y}(t) = \lambda(t)$ and $\lambda_{X,Y} = \lambda$ for simplicity.

- product copula: $\lambda(t) = 1 - t$ and $\lambda = \mathbf{0}$.
- upper Fréchet-Hoeffding bound: $\lambda(t) = 1$ and $\lambda = 1$.
- lower Fréchet-Hoeffding bound: $\lambda(t) = 0$ for $t \ge 1/2$ and $\lambda = \mathbf{0}$.
- normal copula C_ρ: $\lambda = \mathbf{0}$.

Do the lower FH bound (complete negative dependence) and the normal copula have the same tail dependence as the product copula (independence)?

	product	lower FH bound	normal with $\rho = 0.5$
$t = 0.95$	0.0500	0.0000	0.2438
$t = 0.99$	0.0100	0.0000	0.1294
$t = 0.995$	0.0050	0.0000	0.0993
$t = 0.999$	0.0010	0.0000	0.0543

Same limit but different asymptotic behavior!

Asymptotic behavior

- product copula: $\lambda(t) = 1 - t$.
- lower Fréchet-Hoeffding bound: $\lambda(t) = 0$.
- How about the normal copula?

Main Theorem

If C_ρ is the normal copula, then

$$\lambda(t) = \sqrt{\frac{(1+\rho)^3}{2\pi(1-\rho)}}\, e^{-\frac{1-\rho}{2(1+\rho)}s^2} \left(s^{-1} - \frac{1+2\rho-\rho^2}{1-\rho} s^{-3} + O(s^{-5}) \right)$$

$$\sim (4\pi)^{-\frac{\rho}{1+\rho}} \sqrt{\frac{(1+\rho)^3}{1-\rho}} (1-t)^{\frac{1-\rho}{1+\rho}} (-\log(1-t))^{-\frac{\rho}{1+\rho}}.$$

Here $t = \Phi(s) = (2\pi)^{-1/2} \int_{-\infty}^{s} \exp(-x^2/2)\, dx$ (distribution function of the standard normal distribution).

References

[1] Hiroki Kondo, Shingo Saito, and Setsuo Taniguchi, *Asymptotic tail dependence of the normal copula*, preprint, available at http://gcoe-mi.jp/english/publish_list/pub_inner/id:3/cid:16/.