OHNO RELATION FOR REGULARIZED MULTIPLE ZETA VALUES

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Abstract. The Ohno relation for multiple zeta values can be formulated as saying that a certain operator, defined for indices, is invariant under taking duals. In this paper, we generalize the Ohno relation to regularized multiple zeta values by showing that, although the suitably generalized operator is not invariant under taking duals, the relation between its values at an index and at its dual index can be written explicitly in terms of the gamma function.

1. Introduction

Multiple zeta values (MZVs) are real numbers defined by

$$\zeta(k_1,\dots,k_d) \coloneqq \sum_{0<m_1<\cdots<m_d} \frac{1}{m_1^{k_1}\cdots m_d^{k_d}},$$

where $k_1,\dots,k_d$ are positive integers with $k_d>1$. A lot of formulas for MZVs are known and the Ohno relation is one of the most famous among such formulas. Put $\mathfrak{h} \coloneqq \mathbb{Q}\langle x,y\rangle$ and $\mathfrak{h}^0 \coloneqq \mathbb{Q}\oplus y\mathfrak{h}x$. Define a linear map $Z:\mathfrak{h}^0\to\mathbb{R}$, a ring homomorphism $\sigma:\mathfrak{h}\to\mathfrak{h}[[T]]$, and an anti-automorphism $\tau:\mathfrak{h}\to\mathfrak{h}$ by

$$Z(yx^{k_1-1}\cdots yx^{k_d-1}) = \zeta(k_1,\dots,k_d),$$

$$\sigma(x)=x, \sigma(y)=y\frac{1}{1-xT},$$

and

$$\tau(x)=y, \tau(y)=x.$$

In this paper, we use T,A,B as parameters of formal power series, and we regard any linear map g on a $\mathbb{Q}$-vector space R as naturally extended to the linear map g on $R\otimes\mathbb{Q}[[T,A,B]]$ by $\sum_{t,a,b\ge0} g(c_{t,a,b}T^tA^aB^b) = \sum_{t,a,b\ge0} g(c_{t,a,b})T^tA^aB^b$. Then $Z\circ\sigma(w)$ is defined for any $w\in\mathfrak{h}^0$ as an element in $\mathbb{R}_T \coloneqq \mathbb{R}[[T]]$.

Theorem 1 (Ohno relation for MZVs, [7])**.** *For $w\in\mathfrak{h}^0$, we have*

$$Z\circ\sigma\circ\tau(w) = Z\circ\sigma(w).$$

Remark 2. The original formulation of the Ohno relation in [7] is described as

$$\sum_{\substack{e_1+\cdots+e_r=m\\ e_1,\dots,e_r\ge0}} \zeta(k_1+e_1,\dots,k_r+e_r) = \sum_{\substack{e_1+\cdots+e_s=m\\ e_1,\dots,e_s\ge0}} \zeta(k'_1+e_1,\dots,k'_s+e_s),$$

where $(k'_1,\dots,k'_s)$ is the dual index of $(k_1,\dots,k_r)$ i.e., $\tau(yx^{k_1-1}\cdots yx^{k_r-1}) = yx^{k'_1-1}\cdots yx^{k'_s-1}$, and it corresponds to the equality of the coefficient of T^m in Theorem 1 in the case $w=yx^{k_1-1}\cdots yx^{k_d-1}$. A formulation similar to the above theorem can be found in [5].

The shuffle product $⧢:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}$ is the bilinear map defined by

$$w ⧢ 1 = 1 ⧢ w = w \quad (w\in\mathfrak{h}),$$

$$u_1w_1 ⧢ u_2w_2 = u_1(w_1 ⧢ u_2w_2) + u_2(u_1w_1 ⧢ w_2) \quad (u_i\in\{x,y\}, w_i\in\mathfrak{h}).$$

It is known that $Z(w_1 ⧢ w_2) = Z(w_1)Z(w_2)$ for $w_1,w_2\in\mathfrak{h}^0$. Let $Z^{⧢}_{X,Y}:\mathfrak{h}\to\mathbb{R}[X,Y]$ be the unique map characterized by $Z^{⧢}_{X,Y}\big|_{\mathfrak{h}^0} = Z$, $Z^{⧢}_{X,Y}(w_1 ⧢ w_2) = Z^{⧢}_{X,Y}(w_1)Z^{⧢}_{X,Y}(w_2)$ for $w_1,w_2\in\mathfrak{h}$, $Z^{⧢}_{X,Y}(x)=X$, and

2010 *Mathematics Subject Classification.* Primary 11M32.
Key words and phrases. Multiple zeta values, Ohno relation, Regularized multiple zeta values.

$Z^{ш}_{X,Y}(y) = Y$. We put $Z^{ш} = Z^{ш}_{0,0}$. Then for any $w \in \mathfrak{h}$, $Z^{ш}_{X,Y}(w)$ becomes a polynomial of X and Y whose coefficients are linear combinations of MZVs, and called a (shuffle) regularized MZV. The term 'shuffle regularized MZVs' often means the values $\zeta^{ш}(k_1,\dots,k_d) = Z^{ш}_{0,0}(yx^{k_1-1}\cdots yx^{k_d-1})$ with $k_1,\dots,k_d \geq 1$, namely the values for words starting with y, but in this paper, we also consider the values for words starting with x. The main theorem of this paper is a generalization of Theorem 1 to regularized MZVs. To state the main theorem, define an $\mathbb{R}$-linear map $\rho : \mathbb{R}[X,Y] \to \mathbb{R}_T[X,Y]$ by the equality

$$\rho(e^{AX+BY}) = \frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)} e^{AX+BY}$$

in $\mathbb{R}_T[X,Y][[A,B]]$. Equivalently, ρ is determined by

$$\rho\left(\frac{X^kY^l}{k!l!}\right) = \sum_{k'=0}^{k}\sum_{l'=0}^{l} c(k-k',l-l')\frac{X^{k'}Y^{l'}}{k'!l'!},$$

where the coefficients $c(-,-) \in \mathbb{R}_T$ are given by

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c(k,l)A^kB^l = \frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)}.$$

Theorem 3 (Main theorem; Ohno relation for regularized MZVs)**.** *For $w \in \mathfrak{h}$, we have*

$$Z^{ш}_{Y,X} \circ \sigma \circ \tau(w) = \rho \circ Z^{ш}_{X,Y} \circ \sigma(w).$$

Example 4. Let $(k_1,\dots,k_d) \in \mathbb{Z}^d_{\geq 1}$ be an admissible index (i.e., $k_d > 1$) and $(k'_1,\dots,k'_r)$ its dual index. Let us look at the case $w = xyx^{k_1-1}\cdots yx^{k_d-1}$ in Theorem 3. By using the notation $\zeta_l(k_1,\dots,k_d) := Z^{ш}(x^l yx^{k_1-1}\cdots yx^{k_d-1})$, we have

$$\begin{aligned} Z^{ш}_{Y,X} \circ \sigma \circ \tau(w) &= \sum_{m=0}^{\infty} T^m \sum_{\substack{e_1+\cdots+e_{r+1}=m \\ e_1,\dots,e_{r+1}\geq 0}} \zeta_0(k'_1+e_1,\dots,k'_r+e_r,1+e_{r+1}) \\ &\quad + X\sum_{m=0}^{\infty} T^m \sum_{\substack{e_1+\cdots+e_r=m \\ e_1,\dots,e_r\geq 0}} \zeta(k'_1+e_1,\dots,k'_r+e_r) \end{aligned}$$

and

$$\begin{aligned} Z^{ш}_{X,Y} \circ \sigma(w) &= \sum_{m=0}^{\infty} T^m \sum_{\substack{e_1+\cdots+e_d=m \\ e_1,\dots,e_d\geq 0}} \zeta_1(k_1+e_1,\dots,k_d+e_d) \\ &\quad + X\sum_{m=0}^{\infty} T^m \sum_{\substack{e_1+\cdots+e_d=m \\ e_1,\dots,e_d\geq 0}} \zeta(k_1+e_1,\dots,k_d+e_d). \end{aligned}$$

Since $\rho(X) = X + \sum_{n=1}^{\infty}\zeta(n+1)T^n$, the equality of the coefficients of $T^mX^1Y^0$ implies the original Ohno relation, and the equality of the coefficients of $T^mX^0Y^0$ implies

$$\begin{aligned} &\sum_{\substack{e_1+\cdots+e_{r+1}=m \\ e_1,\dots,e_{r+1}\geq 0}} \zeta_0(k'_1+e_1,\dots,k'_r+e_r,1+e_{r+1}) \\ &= \sum_{\substack{e_1+\cdots+e_d=m \\ e_1,\dots,e_d\geq 0}} \zeta_1(k_1+e_1,\dots,k_d+e_d) + \sum_{\substack{e_1+\cdots+e_d+n=m \\ e_1,\dots,e_d\geq 0, n\geq 1}} \zeta(n+1)\zeta(k_1+e_1,\dots,k_d+e_d). \end{aligned}$$

The following is an equivalent formulation of Theorem 3:

Theorem 5 (Equivalent formulation of Theorem 3)**.** *For $w \in \mathfrak{h}^0$, we have*

$$Z^{ш} \circ \sigma \circ \tau\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right) = Z^{ш} \circ \sigma\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right) \times \frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)}. \tag{1.1}$$

2. Proof of the equivalence of Theorems 3 and 5

Put $\bar{\sigma}=\tau\circ\sigma\circ\tau$. Then since $Z^{ш}_{Y,X}=Z^{ш}_{X,Y}\circ\tau$, we have $Z^{ш}_{Y,X}\circ\sigma\circ\tau=Z^{ш}_{X,Y}\circ\bar{\sigma}$, which also implies that $Z^{ш}\circ\sigma\circ\tau=Z^{ш}\circ\bar{\sigma}$. Note that any element of $\mathfrak{h}$ can be written as a linear combination of the terms $x^a w y^b$ with $w\in\mathfrak{h}^0$ and $a,b\in\mathbb{Z}_{\geq 0}$. Thus Theorem 3 is equivalent to the statement that

$$Z^{ш}_{X,Y}\circ\bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right)=\rho\circ Z^{ш}_{X,Y}\circ\sigma\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right) \tag{2.1}$$

for all $w\in\mathfrak{h}^0$. Define a linear map $\mathrm{reg}_{ш}:\mathfrak{h}\to\mathfrak{h}^0$ by

$$\mathrm{reg}_{ш}(w ш x^a ш y^b)=w\delta_{a,0}\delta_{b,0}\quad (w\in\mathfrak{h}^0).$$

Then $Z^{ш}=Z\circ\mathrm{reg}_{ш}$. For $a,b\in\mathbb{Z}_{\geq 0}$, define a linear map $D_{a,b}:\mathfrak{h}\to\mathfrak{h}$ by

$$D_{a,b}(u_1\cdots u_m)=\begin{cases}u_{a+1}\cdots u_{m-b} & m\geq a+b, u_1=\cdots=u_a=x, u_{m-b+1}=\cdots=u_m=y\\ 0 & \text{otherwise,}\end{cases}$$

where $u_1,\dots,u_m\in\{x,y\}$. Then we have

$$w=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\mathrm{reg}_{ш}(D_{a,b}(w)) ш x^a ш y^b \tag{2.2}$$

for $w\in\mathfrak{h}$ as a special case of the following lemma.

Lemma 6 ([8, Lemma 3.2.4 and equation (3.2.20)])**.** *Let Σ be a set, and $\mathcal{A}\coloneqq\mathbb{Q}\langle\Sigma\rangle$ the free non-commutative polynomial ring generated by Σ. Define a linear map $c:\mathcal{A}\to\mathbb{Q}$ by $c(u_1\cdots u_k)=\delta_{k,0}$, where $u_1,\dots,u_k\in\Sigma$. Let A and B be disjoint subsets of Σ. Put*

$$\mathcal{A}^0\coloneqq\sum_{k=0}^{\infty}\sum_{\substack{u_1,\dots,u_k\in\Sigma\\ u_1\notin B,u_k\notin A}}u_1\cdots u_k\mathbb{Q}\subset\mathcal{A}$$

and define a linear map $s:\mathcal{A}^0\otimes\mathbb{Q}\langle A\rangle\otimes\mathbb{Q}\langle B\rangle\to\mathcal{A}$ by $s(w_1\otimes w_2\otimes w_3)=w_1 ш w_2 ш w_3$. Then s is bijective. Furthermore, for $(a_1,\dots,a_r)\in A^r$, $(b_1,\dots,b_l)\in B^l$, and $w\in\mathcal{A}^0$, we have

$$b_1\cdots b_l w a_r\cdots a_1=\sum_{i=0}^{l}\sum_{j=0}^{r}b_1\cdots b_i ш a_j\cdots a_1 ш \mathrm{reg}_A^B\left(b_{i+1}\cdots b_l w a_r\cdots a_{j+1}\right),$$

where reg_A^B is the map from $\mathcal{A}$ to $\mathcal{A}^0$ defined by $\mathrm{reg}_A^B=(\mathrm{id}\otimes c\otimes c)\circ s^{-1}$.

By (2.2), we have

$$Z^{ш}_{X,Y}(w)=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{X^aY^b}{a!b!}Z^{ш}(D_{a,b}(w))\quad (w\in\mathfrak{h}).$$

Thus the left-hand side of (2.1) is

$$Z^{ш}_{X,Y}\circ\bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right)=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{X^aY^b}{a!b!}Z^{ш}\circ D_{a,b}\circ\bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right).$$

Let us calculate $D_{a,b}\circ\bar{\sigma}(\frac{1}{1-xA}w\frac{1}{1-yB})$. By definition,

$$\bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{1}{1-yT}x\right)^k\bar{\sigma}(w)y^lA^kB^l.$$

Since $\bar{\sigma}(w)\in\mathfrak{h}^0$, we have

$$D_{a,b}\left(\left(\frac{1}{1-yT}x\right)^k\bar{\sigma}(w)y^l\right)=0$$

if $k < a$ or $l < b$. Thus

$$\begin{aligned} D_{a,b} \circ \bar{\sigma}\left(\frac{1}{1-xA} w \frac{1}{1-yB}\right) &= \sum_{k=a}^{\infty}\sum_{l=b}^{\infty} D_{a,b}\left(\left(\frac{1}{1-yT}x\right)^{k} \bar{\sigma}(w) y^{l}\right) A^{k}B^{l} \\ &= A^{a}B^{b}D_{a,b}\left(\left(\frac{1}{1-yT}x\right)^{a}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{1}{1-yT}x\right)^{k}\bar{\sigma}(w)y^{l}A^{k}B^{l}\right)y^{b}\right) \\ &= A^{a}B^{b}D_{a,b}\left(\left(\frac{1}{1-yT}x\right)^{a}\bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right)y^{b}\right) \\ &= A^{a}B^{b}\bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right) \quad \text{(by the definition of } D_{a,b}\text{)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} Z^{ш}_{X,Y} \circ \bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right) &= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{X^{a}Y^{b}}{a!b!}A^{a}B^{b} \times Z^{ш} \circ \bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right) \\ &= e^{AX+BY} \times Z^{ш} \circ \bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right). \end{aligned}$$

Similarly the right-hand side of (2.1) is

$$\begin{aligned} &\rho \circ Z^{ш}_{X,Y} \circ \sigma\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right) \\ &= \rho\left(e^{AX+BY} \times Z^{ш} \circ \sigma\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right)\right) \\ &= e^{AX+BY} \times Z^{ш} \circ \sigma\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right) \times \frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)}. \end{aligned}$$

Thus (2.1) is equivalent to (1.1).

3. Symmetric harmonic product

Put $\mathfrak{h}' := y\mathfrak{h} \oplus x\mathfrak{h}$. Let $e_0 = x$, $e_1 = -y$. Define the *symmetric harmonic product* $\tilde{*} : \mathfrak{h}' \times \mathfrak{h}' \to \mathfrak{h}'$ as the bilinear map satisfying

$$\begin{aligned} e_a \,\tilde{*}\, e_{b_1} \cdots e_{b_n} &= e_{b_1} \cdots e_{b_n} \,\tilde{*}\, e_a = e_{ab_1} \cdots e_{ab_n}, \\ e_a w_1 \,\tilde{*}\, e_b w_2 &= e_{ab}(w_1 \,\tilde{*}\, e_b w_2) + e_{ab}(e_a w_1 \,\tilde{*}\, w_2) - e_{ab} e_0 (w_1 \,\tilde{*}\, w_2) \quad (w_1, w_2 \in \mathfrak{h}'), \end{aligned}$$

where $a, b, b_1, \ldots, b_n \in \{0,1\}$. Then $\tilde{*}$ has a following combinatorial description which is similar to the one for $*$ given in [3]:

$$e_{a_0} \cdots e_{a_m} \,\tilde{*}\, e_{b_0} \cdots e_{b_n} = \sum_{\mathbf{p}=(p_0,\ldots,p_{m+n})} (-1)^{\#\{i \mid p_i \in (1/2+\mathbb{Z})^2\}} e_{c(p_0)} \cdots e_{c(p_{m+n})},$$

where $\mathbf{p} = (p_0, \ldots, p_{m+n})$ runs over all sequences of elements of $(\frac{1}{2}\mathbb{Z})^2$ such that $p_0 = (0,0)$, $p_{m+n} = (m,n)$, $p_{i+1} - p_i \in \{(0,1),(1,0),(\frac{1}{2},\frac{1}{2})\}$, and $\{p_i, p_{i+1}\} \not\subset (\frac{1}{2}+\mathbb{Z})^2$ for all $0 \le i < m+n$, and $c(p_i)$ is defined by

$$c(p_i) = \begin{cases} a_x b_y & p_i = (x,y) \in \mathbb{Z}^2, \\ 0 & p_i \in (\frac{1}{2}+\mathbb{Z})^2. \end{cases}$$

By this combinatorial description, we can see that the symmetric harmonic product $\tilde{*}$ is compatible with the reversal operation, i.e.,

$$\overleftarrow{w_1} \,\tilde{*}\, \overleftarrow{w_2} = \overleftarrow{w_1 \,\tilde{*}\, w_2} \quad (w_1, w_2 \in \mathfrak{h}'), \tag{3.1}$$

where $\overleftarrow{w}$ is the reversal word of w. By definition, we have

$$w_1 e_1 \,\tilde{*}\, w_2 e_1 = (w_1 * w_2) e_1 \quad (w_1, w_2 \in \mathfrak{h}), \tag{3.2}$$

where $*:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}$ is the usual harmonic product defined in [3]. Note that by the correspondence

$$yx^{k_1-1}\cdots yx^{k_d-1}\longleftrightarrow(k_1,\dots,k_d),$$

the product $*$ on $\mathbb{Q}+y\mathfrak{h}$ corresponds to the harmonic product on indices (see [3] for details).

Thus we have

$$xw_1\,\tilde{*}\,w_2=w_1\,\tilde{*}\,xw_2=x(w_1\,\tilde{*}\,w_2)\quad(w_1,w_2\in\mathfrak{h}'),\tag{3.3}$$

$$yw_1\,\tilde{*}\,yw_2=y(w_1\,\tilde{*}\,yw_2)+y(yw_1\,\tilde{*}\,w_2)+yx(w_1\,\tilde{*}\,w_2)\quad(w_1,w_2\in\mathfrak{h}').\tag{3.4}$$

By (3.1), (3.3), and (3.4), we have

$$w_1x\,\tilde{*}\,w_2=w_1\,\tilde{*}\,w_2x=(w_1\,\tilde{*}\,w_2)x\quad(w_1,w_2\in\mathfrak{h}'),\tag{3.5}$$

$$w_1y\,\tilde{*}\,w_2y=(w_1\,\tilde{*}\,w_2y)y+(w_1y\,\tilde{*}\,w_2)y+(w_1\,\tilde{*}\,w_2)xy\quad(w_1,w_2\in\mathfrak{h}').\tag{3.6}$$

Furthermore, putting $z=x+y$, we also have

$$w_1z\,\tilde{*}\,w_2z=-(w_1*w_2)z\quad(w_1,w_2\in\mathfrak{h})\tag{3.7}$$

since

$$\begin{aligned}w_1z\,\tilde{*}\,w_2z&=w_1y\,\tilde{*}\,w_2y+(w_1\,\tilde{*}\,w_2y)x+(w_1z\,\tilde{*}\,w_2)x=(w_1\,\tilde{*}\,w_2y)z+(w_1z\,\tilde{*}\,w_2)z\\&=R_y^{-1}(w_1y\,\tilde{*}\,w_2y)z=-(w_1*w_2)z,\end{aligned}$$

where $R_y:\mathfrak{h}\to\mathfrak{h}y$ is defined by $R_y(w)=wy$.

4. Proof of the main theorem assuming a few propositions

In this section, we give a proof of Theorem 5, assuming some propositions which will be proved in later sections. Since $\mathfrak{h}^0\coloneqq\mathbb{Q}\oplus y\mathfrak{h}x$, Theorem 5 is reduced to the two claims

$$Z^{\sqcup\!\sqcup}\circ\bar{\sigma}\left(\frac{1}{1-xA}\frac{1}{1-yB}\right)=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-xA}\frac{1}{1-yB}\right)\times\frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)}\tag{4.1}$$

and

$$\forall w\in y\mathfrak{h}x,\quad Z^{\sqcup\!\sqcup}\circ\bar{\sigma}\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right)=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-xA}w\frac{1}{1-yB}\right)\times\frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)}.\tag{4.2}$$

Then (4.2) is equivalent to

$$\forall w\in y\mathfrak{h}x,\quad Z^{\sqcup\!\sqcup}\circ\bar{\sigma}\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)\times\frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)}\tag{4.3}$$

since

$$\begin{aligned}\left\{\frac{1}{1-xA}w\frac{1}{1-yB}\,\middle|\,w\in y\hat{\mathfrak{h}}x\right\}&=\{u\in\hat{\mathfrak{h}}\mid(1-xA)u(1-yB)\in y\hat{\mathfrak{h}}x\}\\&=\{u\in\hat{\mathfrak{h}}\mid(1-zA)u(1-zB)\in y\hat{\mathfrak{h}}x\}\\&=\left\{\frac{1}{1-zA}w\frac{1}{1-zB}\,\middle|\,w\in y\hat{\mathfrak{h}}x\right\},\end{aligned}$$

where $\hat{\mathfrak{h}}=\mathfrak{h}[[A,B]]$.

Let φ be the automorphism of $\mathfrak{h}$ defined by $\varphi(x)=z$, $\varphi(y)=-y$. Note that $\varphi\circ\varphi=\mathrm{id}$. Define $H:\mathfrak{h}\to\mathbb{R}[[A,B]]$ by

$$H(w)=Z^{\sqcup\!\sqcup}\circ\varphi\left(\frac{1}{1-xA}w\frac{1}{1-xB}\right).$$

Our main theorem can be proved by combining the following propositions which will be proved in later sections:

Proposition 7. *For $w\in\mathfrak{h}$, we have*

$$\bar{\sigma}(w)=\sigma(w)+\varphi\left(\frac{yT}{1+yT}z\,\tilde{*}\,\varphi(\sigma(w))\right). \tag{4.4}$$

Proposition 8. *For $w_1, w_2 \in y\mathfrak{h}z$, we have*

$$H(w_1\,\tilde{*}\,w_2)=-H(w_1)H(w_2)\frac{\sin\pi(A-B)}{\pi}.$$

Proposition 9. *We have*

$$-H\left(\frac{yT}{1+yT}z\right)=\frac{\pi}{\sin\pi(A-B)}\left(\frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)}-1\right). \tag{4.5}$$

Before proceeding to the proof of the main theorem, let us introduce the following useful formulas which will sometimes be used in the remainder of this paper.

Lemma 10. *Put $\mathcal{R}:=\sum_{v\in\{x,y\}}\sum_{S\in\{T,A,B\}}\mathbb{Q}Sv$. For $P,Q\in\mathcal{R}$, $w_0,\dots,w_m\in\mathfrak{h}[[T,A,B]]$, and $u_1,\dots,u_m\in\{x,y\}$, we have*

$$\frac{1}{1-P}\sqcup\!\sqcup\, w_0u_1w_1\cdots u_mw_m=\left(\frac{1}{1-P}\sqcup\!\sqcup\, w_0\right)u_1\left(\frac{1}{1-P}\sqcup\!\sqcup\, w_1\right)\cdots u_m\left(\frac{1}{1-P}\sqcup\!\sqcup\, w_m\right)$$

and

$$\frac{1}{1-P}\sqcup\!\sqcup\,\frac{1}{1-Q}=\frac{1}{1-P-Q}.$$

We omit the proof of this lemma since it is almost obvious.

Proof of Theorem 5 assuming Propositions 7, 8, and 9. It is enough to prove (4.3) and (4.1).

To prove (4.3), we take $w\in y\mathfrak{h}x$ and set $u:=(1-zA)\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)(1-zB)$. Then

$$\varphi(u)\in y\mathfrak{h}z[[T,A,B]] \tag{4.6}$$

since

$$\begin{aligned}u&=\sigma\left((1-zA+yxAT)\frac{1}{1-zA}w\frac{1}{1-zB}(1-zB+yxBT)\right)\\&=\sigma\left(\left(1+yxT\frac{A}{1-zA}\right)w\left(1+yxT\frac{B}{1-zB}\right)\right)\\&\in\sigma\left(y\mathfrak{h}x[[T,A,B]]\right)\subset y\mathfrak{h}x[[T,A,B]].\end{aligned}$$

Now (4.3) follows from the following calculation:

$$\begin{aligned}&Z^{\sqcup\!\sqcup}\circ\bar{\sigma}\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)\\&=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)+Z^{\sqcup\!\sqcup}\circ\varphi\left(\frac{yT}{1+yT}z\,\tilde{*}\,\varphi\left(\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)\right)\right)\quad\text{(by Proposition 7)}\\&=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)+Z^{\sqcup\!\sqcup}\circ\varphi\left(\frac{yT}{1+yT}z\,\tilde{*}\,\frac{1}{1-xA}\varphi(u)\frac{1}{1-xB}\right)\\&=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)+H\left(\frac{yT}{1+yT}z\,\tilde{*}\,\varphi(u)\right)\quad\text{(by (3.3) and (3.5))}\\&=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)-H\left(\frac{yT}{1+yT}z\right)H(\varphi(u))\frac{\sin\pi(A-B)}{\pi}\quad\text{(by (4.6) and Proposition 8)}\\&=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)\left(1-H\left(\frac{yT}{1+yT}z\right)\frac{\sin\pi(A-B)}{\pi}\right)\\&=Z^{\sqcup\!\sqcup}\circ\sigma\left(\frac{1}{1-zA}w\frac{1}{1-zB}\right)\cdot\frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+B)\Gamma(1-T+A)}\quad\text{(by Proposition 9),}\end{aligned}$$

We now prove (4.1). Since

$$\begin{aligned}
&\bar{\sigma}\left(\frac{1}{1-xA}\frac{1}{1-yB}\right)\\
&=\frac{1}{1-\frac{1}{1-yT}xA}\frac{1}{1-yB}\\
&=\frac{1}{1-yT-xA}(1-yT)\frac{1}{1-yB}\\
&=\frac{1}{1-yT-xA}+\frac{1}{1-yT-xA}y\frac{B-T}{1-yB}\\
&=\frac{1}{1-xA}ш\frac{1}{1-yT}+\frac{1}{1-xA}ш\frac{1}{1-yB}ш\frac{1}{1+y(B-T)}y\frac{B-T}{1+xA}\quad\text{(by Lemma 10)},
\end{aligned}$$

we have

$$\begin{aligned}
Z^{ш}\circ\bar{\sigma}\left(\frac{1}{1-xA}\frac{1}{1-yB}\right)&=Z^{ш}\left(1+\frac{1}{1+y(B-T)}y\frac{B-T}{1+xA}\right)\\
&=1-A(B-T)Z^{ш}\left(\frac{1}{1+y(B-T)}yx\frac{1}{1+xA}\right)\\
&=1-\sum_{a,b\geq1}\zeta(\overbrace{1,\dots,1}^{a-1},b+1)(T-B)^a(-A)^b\\
&=\frac{\Gamma(1+A)\Gamma(1-T+B)}{\Gamma(1+A+B-T)}.
\end{aligned}$$

(see [1, equation (10)] for the last equality). Similarly, we also have

$$Z^{ш}\circ\sigma\left(\frac{1}{1-xA}\frac{1}{1-yB}\right)=\frac{\Gamma(1+B)\Gamma(1-T+A)}{\Gamma(1+A+B-T)}.$$

Thus (4.1) is also proved. □

5. Proof of Proposition 7

In this section, we prove Proposition 7. For $m\in\mathbb{Z}$, define $\sigma_m:\mathfrak{h}\to\mathfrak{h}$ by $\sigma(w)=\sum_{m=0}^{\infty}T^m\sigma_m(w)$ and $\sigma_n(w)=0$ for $n<0$. Furthermore, we put $\bar{\sigma}_m:=\tau\circ\sigma_m\circ\tau$. Let $\diamond:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}$ be the bilinear map defined in [2]. It satisfies $\varphi(w_1)*\varphi(w_2)=\varphi(w_1\diamond w_2)$, $yw_1\diamond yw_2=y(yw_1\diamond w_2)-y(w_1\diamond xw_2)$, and $yw_1\diamond xw_2=x(yw_1\diamond w_2)+y(w_1\diamond xw_2)$. For $n\in\mathbb{Z}$ and $w\in\mathfrak{h}$, we put $f_n(w):=y^n\diamond w-(y^{n-1}\diamond w)y$, where we mean $y^m=0$ for $m<0$, i.e.,

$$f_n(w)=\begin{cases}0 & n<0,\\ w & n=0,\\ y^n\diamond w-(y^{n-1}\diamond w)y & n\geq1.\end{cases}$$

For $n\geq1$, since

$$f_n(w)=(-1)^n\varphi\left(y^n*\varphi(w)-(y^{n-1}*\varphi(w))y\right)$$

and

$$\begin{aligned}
&\left(y^n*\varphi(w)-(y^{n-1}*\varphi(w))y\right)y\\
&=-y^{n+1}\tilde{*}\varphi(w)y+(y^n\tilde{*}\varphi(w)y)y\quad\text{(by (3.2))}\\
&=-\left(y^{n+1}\tilde{*}\varphi(w)\right)y-(y^n\tilde{*}\varphi(w))xy\quad\text{(by (3.6))}\\
&=-(y^nz\tilde{*}\varphi(w))y\quad\text{(by (3.5))},
\end{aligned}$$

we have

$$f_n(w)=(-1)^{n+1}\varphi\left(y^nz\tilde{*}\varphi(w)\right).\tag{5.1}$$

Lemma 11. *For $n \in \mathbb{Z}$ and $w \in \mathfrak{h}$, we have*

$$\begin{aligned} f_n(xw) &= x f_n(w) + y f_{n-1}(xw), \\ f_n(yw) &= y f_n(w) - y f_{n-1}(xw). \end{aligned}$$

Proof. The lemma follows from $yw_1 \diamond xw_2 = x(yw_1 \diamond w_2) + y(w_1 \diamond xw_2)$ and $yw_1 \diamond yw_2 = y(yw_1 \diamond w_2) - y(w_1 \diamond xw_2)$. □

Lemma 12. *For $n \geq 0$ and $w \in \mathfrak{h}$, we have*

$$\sum_{j=0}^{n} f_j(yx^{n-j}w) = y f_n(w).$$

Proof. By Lemma 11, we have

$$\sum_{j=0}^{n} f_j(yx^{n-j}w) = y \sum_{j=0}^{n} f_j(x^{n-j}w) - y \sum_{j=1}^{n} f_{j-1}(x^{n-j+1}w) = y f_n(w). \qquad \square$$

Lemma 13. *For $m \geq 0$ and $w \in \mathfrak{h}$, we have*

$$\begin{aligned} \bar{\sigma}_m(xw) &= y\bar{\sigma}_{m-1}(xw) + x\bar{\sigma}_m(w), \\ \bar{\sigma}_m(yw) &= y\bar{\sigma}_m(w), \\ \sum_{j=0}^{m} f_j(\sigma_{m-j}(xw)) &= y \sum_{j=0}^{m-1} f_j(\sigma_{m-1-j}(xw)) + x \sum_{j=0}^{m} f_j(\sigma_{m-j}(w)), \\ \sum_{j=0}^{m} f_j(\sigma_{m-j}(yw)) &= y \sum_{j=0}^{m} f_j(\sigma_{m-j}(w)). \end{aligned}$$

Proof. The first two identities are obvious. The last two identities follow from the following calculation:

$$\begin{aligned} \sum_{j=0}^{m} f_j(\sigma_{m-j}(xw)) &= \sum_{j=0}^{m} f_j(x\sigma_{m-j}(w)) \\ &= x \sum_{j=0}^{m} f_j(\sigma_{m-j}(w)) + y \sum_{j=1}^{m} f_{j-1}(x\sigma_{m-j}(w)) \quad \text{(by Lemma 11)}, \\ \sum_{j=0}^{m} f_j(\sigma_{m-j}(yw)) &= \sum_{j=0}^{m} \sum_{i=0}^{m-j} f_j(yx^i\sigma_{m-j-i}(w)) \\ &= \sum_{n=0}^{m} \sum_{j=0}^{n} f_j(yx^{n-j}\sigma_{m-n}(w)) \qquad (i+j=n) \\ &= \sum_{n=0}^{m} y f_n(\sigma_{m-n}(w)) \qquad \text{(by Lemma 12)}. \end{aligned} \qquad \square$$

Lemma 14. *For $m \geq 0$ and $w \in \mathfrak{h}$, we have*

$$\bar{\sigma}_m(w) = \sum_{j=0}^{m} f_j(\sigma_{m-j}(w)).$$

Proof. Let us prove this lemma by induction on m and on the length of w. The case $m = 0$ or $w = 1$ is obvious. Furthermore, both sides satisfy the same recurrence formula by Lemma 13. Thus the lemma is proved. □

Now, Proposition 7 follows from the following calculation:

$$\begin{aligned}\bar{\sigma}(w) &= \sum_{m=0}^{\infty} T^m \cdot \bar{\sigma}_m(w) \\ &= \sum_{m=0}^{\infty} T^m \sum_{j=0}^{m} f_j(\sigma_{m-j}(w)) \quad \text{(by Lemma 14)} \\ &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} T^{n+j} f_j(\sigma_n(w)) \quad (n = m-j) \\ &= \sum_{j=0}^{\infty} T^j f_j(\sigma(w)) \\ &= \sigma(w) - \sum_{j=1}^{\infty} (-T)^j \varphi\left(y^j z \,\tilde{*}\, \varphi(\sigma(w))\right) \quad \text{(by (5.1))} \\ &= \sigma(w) + \varphi\left(\frac{yT}{1+yT} z \,\tilde{*}\, \varphi(\sigma(w))\right).\end{aligned}$$

6. Proof of Proposition 8

In this section, we prove Proposition 8 by using the Kawashima relation. For $n \geq 1$, put $z_n := yx^{n-1}$. Define a bilinear map $\circledast : \mathfrak{h}y\mathfrak{h} \times \mathfrak{h}y\mathfrak{h} \to \mathfrak{h}y\mathfrak{h}$ by

$$w_1 z_m \circledast w_2 z_n = (w_1 * w_2) z_{m+n}.$$

For $w \in y\mathfrak{h}$ and indeterminate T, define $K(w;T) \in \mathbb{R}[[T]]$ by

$$K(w;T) := \sum_{m=1}^{\infty} Z(y^m \circledast \varphi(w)) T^m.$$

Then the Kawashima relation can be stated as follows:

Theorem 15 (Kawashima relation, [6])**.** *For $w_1, w_2 \in y\mathfrak{h}$, we have*

$$K(w_1 * w_2; T) = K(w_1;T) K(w_2;T).$$

We put $\Gamma_1(T) := \exp(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n}(-T)^n) \in \mathbb{R}[[T]]$ and $\mathfrak{h}^1 = \mathbb{Q} \oplus y\mathfrak{h}$. Then $\mathfrak{h}^1$ is closed under the harmonic product $*$. Define $Z^* : \mathfrak{h}^1 \to \mathbb{R}$ by the properties $Z^*(w_1 * w_2) = Z^*(w_1)Z^*(w_2)$, $Z^*(w) = Z(w)$, and $Z^*(y) = 0$, where $w_1, w_2 \in \mathfrak{h}^1$, and $w \in \mathfrak{h}^0$. Then the regularization theorem can be stated as follows:

Theorem 16 (Regularization theorem, [5])**.** *For $w \in \mathfrak{h}^0$, we have*

$$Z^{\sqcup\!\sqcup}\left(w \frac{1}{1-yT}\right) = \Gamma_1(T) Z^*\left(w \frac{1}{1-yT}\right).$$

Corollary 17. *Let $L \in \mathbb{Q}T$. If $w \in \mathfrak{h}^0[[T]](1-yL)^{-1}$, then*

$$Z^{\sqcup\!\sqcup}(w) = \Gamma_1(L) Z^*(w).$$

Lemma 18. *For $u \in y\mathfrak{h}$, we have*

$$K(u;T) = \frac{\sin(\pi T)}{\pi} Z^{\sqcup\!\sqcup}\left(\varphi(u) x \frac{1}{1+zT}\right).$$

Proof. It is enough to consider the case where $\varphi(u) = vz_k$ for some $v \in \mathfrak{h}^1$ and $k \in \mathbb{Z}_{\geq 1}$. We have

$$\begin{aligned}K(u;T) &= TZ\left(\frac{1}{1-yT} y \circledast vz_k\right) \\ &= TZ\left(\left(\frac{1}{1-yT} * v\right) z_{k+1}\right).\end{aligned}$$

Note that for any formal power series w_1, w_2, and w_3 of T whose coefficients are in $\mathfrak{h}^1$, $\bigoplus_{m=1}^{\infty}\mathbb{Q}z_m$, and $\mathfrak{h}$ respectively, we have

$$\frac{1}{1-yT}*w_1w_2w_3=\left(\frac{1}{1-yT}*w_1\right)w_2(1+xT)\left(\frac{1}{1-yT}*w_3\right) \tag{6.1}$$

since the combinatorial description of the harmonic product between indices shows

$$y^N*w_1w_2w_3=\sum_{N_1+N_2=N}\left(y^{N_1}*w_1\right)w_2\left(y^{N_2}*w_3\right)+\sum_{N_1+1+N_2=N}\left(y^{N_1}*w_1\right)w_2x\left(y^{N_2}*w_3\right).$$

Note that the harmonic inverse of $\frac{1}{1-yT}$ is given by

$$1-yT+yzT^2-yz^2T^2+\cdots=1-yT(1+zT)^{-1}.$$

Thus, by applying (6.1) to $w_1=v$, $w_2=z_{k+1}(1+xT)^{-1}$, and $w_3=1-yT(1+zT)^{-1}$, we obtain

$$\frac{1}{1-yT}*vz_{k+1}\frac{1}{1+zT}=\left(\frac{1}{1-yT}*v\right)z_{k+1},$$

and thus

$$\begin{aligned}K(u;T)&=TZ\left(\frac{1}{1-yT}*vz_{k+1}\frac{1}{1+zT}\right)\\&=TZ^*\left(\frac{1}{1-yT}\right)Z^*\left(vz_{k+1}\frac{1}{1+zT}\right).\end{aligned}$$

Here since

$$\frac{1}{1-yT}\in\mathfrak{h}^0[[T]]\frac{1}{1-yT}\quad\text{and}\quad vz_{k+1}\frac{1}{1+zT}\in\mathfrak{h}^0[[T]]\frac{1}{1+yT},$$

we can apply Corollary 17, and we have

$$\begin{aligned}K(u;T)&=T\Gamma_1(T)^{-1}Z^{\sqcup\!\sqcup}\left(\frac{1}{1-yT}\right)\Gamma_1(-T)^{-1}Z^{\sqcup\!\sqcup}\left(vz_{k+1}\frac{1}{1+zT}\right)\\&=\frac{\sin(\pi T)}{\pi}Z^{\sqcup\!\sqcup}\left(\varphi(u)x\frac{1}{1+zT}\right).\end{aligned}$$

□

Define a linear map $\alpha_A:\mathfrak{h}\to\mathfrak{h}[[A]]$ by

$$\alpha_A(1)=1,\ \alpha_A(vw)=v\left(\frac{1}{1+xA}\sqcup\!\sqcup\, w\right)\quad(v\in\{x,y\},w\in\mathfrak{h}).$$

Lemma 19. *$\alpha_A(w_1*w_2)=\alpha_A(w_1)*\alpha_A(w_2)$ for $w_1,w_2\in\mathfrak{h}^1$.*

Proof. We prove this lemma based on the theory of quasi-symmetric functions in [4, Theorem 2.2]. Let $\mathfrak{P}$ be the inverse limit of $\mathbb{Z}[t_1^{-1},\dots,t_n^{-1}]$ and define a linear map $\psi:\mathfrak{h}^1\to\mathfrak{P}$ by $\psi(yx^{k_1-1}\cdots yx^{k_d-1})=\sum_{0<m_1<\cdots<m_d}t_{m_1}^{-k_1}\cdots t_{m_d}^{-k_d}$. Then ψ is injective and $\psi(w_1*w_2)=\psi(w_1)\psi(w_2)$ for all $w_1,w_2\in\mathfrak{h}^1$ (see [4, Theorem 2.2]). Define the continuous ring automorphism D_A of $\mathfrak{P}[[A]]$ by $D_A(A)=A$ and $D_A(t_m^{-1})=(t_m+A)^{-1}=\sum_{e=0}^{\infty}t_m^{-1-e}(-A)^e$. Then for any $u=yx^{k_1-1}\cdots yx^{k_d-1}\in\mathfrak{h}^1$, we have $\psi(\alpha_A(u))=D_A(\psi(u))$ since

$$\begin{aligned}\psi(\alpha_A(u))&=\sum_{e_1,\dots,e_d\geq0}\psi(yx^{k_1+e_1-1}\cdots yx^{k_d+e_d-1})\prod_{j=1}^{d}\binom{k_j+e_j-1}{e_j}(-A)^{e_j}\\&=\sum_{0<m_1<\cdots<m_d}\prod_{j=1}^{d}\left(\sum_{e_j=0}^{\infty}t_{m_j}^{-k_j-e_j}\binom{k_j+e_j-1}{e_j}(-A)^{e_j}\right)\\&=\sum_{0<m_1<\cdots<m_d}\prod_{j=1}^{d}(t_{m_j}+A)^{-k_j}=D_A(\psi(u)).\end{aligned}$$

Thus for any $w_1,w_2\in\mathfrak{h}^1$, $\psi(\alpha_A(w_1*w_2))=\psi(\alpha_A(w_1)*\alpha_A(w_2))$. Since ψ is injective, the lemma is proved. □

Now we can show Proposition 8.

Proof of Proposition 8. For any $w = w'z \in y\mathfrak{h}z$, we have

$$\begin{aligned}
H(w) &= Z^{ш} \circ \varphi\left(\frac{1}{1-xA}w'z\frac{1}{1-xB}\right) \\
&= Z^{ш} \circ \varphi\left(\frac{1}{1-xA} ш \alpha_A(w')z\frac{1}{1+x(A-B)}\right) \quad \text{(by Lemma 10)} \\
&= Z^{ш} \circ \varphi\left(\alpha_A(w')z\frac{1}{1+x(A-B)}\right) \quad \text{(since } \varphi \text{ is a } ш\text{-homomorphism)} \\
&= Z^{ш}\left(\varphi(\alpha_A(w'))x\frac{1}{1+z(A-B)}\right) \\
&= \frac{\pi}{\sin(\pi(A-B))}K(\alpha_A(w');A-B) \quad \text{(by Lemma 18)}.
\end{aligned}$$

Thus for $u = u'z, v = v'z \in y\mathfrak{h}z$, we have

$$\begin{aligned}
H(u \mathbin{\tilde{*}} v) &= -H((u' * v')z) \quad \text{(by (3.7))} \\
&= -\frac{\pi}{\sin(\pi(A-B))}K(\alpha_A(u' * v');A-B) \\
&= -\frac{\pi}{\sin(\pi(A-B))}K(\alpha_A(u') * \alpha_A(v');A-B) \quad \text{(by Lemma 19)} \\
&= -\frac{\pi}{\sin(\pi(A-B))}K(\alpha_A(u');A-B)K(\alpha_A(v');A-B) \quad \text{(by Theorem 15)} \\
&= -\frac{\sin(\pi(A-B))}{\pi}H(u)H(v).
\end{aligned}$$

□

7. Proof of Proposition 9

The purpose of this section is to prove Proposition 9. The left-hand side of Proposition 9 is equal to

$$-H\left(\frac{yT}{1+yT}z\right) = Z^{ш}\left(\frac{1}{1-zA}\frac{yT}{1-yT}x\frac{1}{1-zB}\right).$$

Here, by Lemma 10, we have

$$\begin{aligned}
&\frac{1}{1-zA}y\frac{1}{1-yT}x\frac{1}{1-zB} \\
&= \frac{1}{1-xA} ш \frac{1}{1-yB} ш \frac{1}{1-yA+yB}y\frac{1}{1+xA+yB-yT}x\frac{1}{1+xA-xB}.
\end{aligned}$$

Thus

$$\begin{aligned}
&-H\left(\frac{yT}{1+yT}z\right) \\
&= TZ\left(\frac{1}{1-yA+yB}y\frac{1}{1+xA+yB-yT}x\frac{1}{1+xA-xB}\right) \\
&= (-\alpha+\beta-\gamma)Z\left(\frac{1}{1-y\alpha}y\frac{1}{1-x\gamma-y\beta}x\frac{1}{1+x\alpha}\right) \quad (\alpha = A-B, \beta = T-B, \gamma = -A).
\end{aligned}$$

Thus it suffices to prove the following identity.

Proposition 20. *We have*

$$\begin{aligned}
&(-\alpha+\beta-\gamma)Z\left(\frac{1}{1-y\alpha}y\frac{1}{1-x\gamma-y\beta}x\frac{1}{1+x\alpha}\right) \\
&= \frac{\pi}{\sin\pi\alpha}\left(\frac{\Gamma(1-\gamma)\Gamma(1-\beta)}{\Gamma(1-\alpha-\gamma)\Gamma(1+\alpha-\beta)} - 1\right).
\end{aligned}$$

Proof. Note that the equality which we want to show is one of formal power series in α, β, γ. By the iterated integral expression of MZVs, we have

$$
\begin{aligned}
&Z\left(\frac{1}{1-y\alpha}y\frac{1}{1-x\gamma-y\beta}x\frac{1}{1+x\alpha}\right)\\
&=\sum_{k,l,m=0}^{\infty} Z\left((\alpha y)^k\,y\,(\gamma x+\beta y)^l\,x(-\alpha x)^m\right)\\
&=\sum_{k,l,m=0}^{\infty}\int_{0<t_a<t_b<1}\left(\frac{1}{k!}\left(\alpha\int_0^{t_a}\frac{ds}{1-s}\right)^k\right)\left(\frac{1}{l!}\left(\gamma\int_{t_a}^{t_b}\frac{ds}{s}+\beta\int_{t_a}^{t_b}\frac{ds}{1-s}\right)^l\right)\left(\frac{1}{m!}\left(-\alpha\int_{t_b}^{1}\frac{ds}{s}\right)^m\right)\frac{dt_a dt_b}{(1-t_a)t_b}\\
&=\int_{0<t_a<t_b<1}\left(\frac{1}{1-t_a}\right)^{\alpha}\left(\frac{t_b}{t_a}\right)^{\gamma}\left(\frac{1-t_a}{1-t_b}\right)^{\beta}\left(\frac{1}{t_b}\right)^{-\alpha}\frac{dt_a dt_b}{(1-t_a)t_b}.
\end{aligned}
$$

Note that

$$\int_{0<t_a<t_b<1}\left(\frac{1}{1-t_a}\right)^{\alpha}\left(\frac{t_b}{t_a}\right)^{\gamma}\left(\frac{1-t_a}{1-t_b}\right)^{\beta}\left(\frac{1}{t_b}\right)^{-\alpha}\frac{dt_a dt_b}{(1-t_a)t_b}$$

converges absolutely as a complex function of α, β, γ in a neighborhood of the origin $(\alpha,\beta,\gamma)=(0,0,0)$. Thus it is enough to show the equality

$$
\begin{aligned}
&(-\alpha+\beta-\gamma)\int_{0<t_a<t_b<1}\left(\frac{1}{1-t_a}\right)^{\alpha}\left(\frac{t_b}{t_a}\right)^{\gamma}\left(\frac{1-t_a}{1-t_b}\right)^{\beta}\left(\frac{1}{t_b}\right)^{-\alpha}\frac{dt_a dt_b}{(1-t_a)t_b}\\
&=\frac{\pi}{\sin\pi\alpha}\left(\frac{\Gamma(1-\gamma)\Gamma(1-\beta)}{\Gamma(1-\alpha-\gamma)\Gamma(1+\alpha-\beta)}-1\right)
\end{aligned}
$$

as complex functions of α,β,γ in a neighborhood of the origin. We denote by $(x)_m$ the Pochhammer symbol $x(x+1)\cdots(x+m-1)$. Then, as a complex function, we have

$$
\begin{aligned}
&\int_{0<t_a<t_b<1}\left(\frac{1}{1-t_a}\right)^{\alpha}\left(\frac{t_b}{t_a}\right)^{\gamma}\left(\frac{1-t_a}{1-t_b}\right)^{\beta}\left(\frac{1}{t_b}\right)^{-\alpha}\frac{dt_a dt_b}{(1-t_a)t_b}\\
&=\int_{0<t_b<1} t_b^{\alpha+\gamma-1}(1-t_b)^{-\beta}\left(\int_{0<t_a<t_b} t_a^{-\gamma}(1-t_a)^{-\alpha+\beta-1}dt_a\right)dt_b\\
&=\int_{0<t_b<1} t_b^{\alpha+\gamma-1}(1-t_b)^{-\beta}\left(\sum_{n=0}^{\infty}\int_{0<t_a<t_b} t_a^{n-\gamma}\frac{(\alpha-\beta+1)_n}{n!}dt_a\right)dt_b\\
&=\int_{0<t_b<1} t_b^{\alpha+\gamma-1}(1-t_b)^{-\beta}\left(\sum_{n=0}^{\infty}\frac{(\alpha-\beta+1)_n}{n!}\frac{t_b^{n-\gamma+1}}{n-\gamma+1}\right)dt_b\\
&=\sum_{n=0}^{\infty}\frac{(\alpha-\beta+1)_n}{n!}\frac{1}{n-\gamma+1}\int_{0<t_b<1} t_b^{\alpha+n}(1-t_b)^{-\beta}dt_b\\
&=\sum_{n=0}^{\infty}\frac{(\alpha-\beta+1)_n}{n!}\frac{1}{n-\gamma+1}\frac{\Gamma(\alpha+n+1)\Gamma(-\beta+1)}{\Gamma(\alpha-\beta+n+2)}\\
&=\frac{\Gamma(\alpha+1)\Gamma(-\beta+1)}{\Gamma(\alpha-\beta+1)}\sum_{n=0}^{\infty}\frac{(\alpha+1)_n}{n!(n-\gamma+1)(\alpha-\beta+n+1)}\\
&=\frac{\Gamma(\alpha+1)\Gamma(-\beta+1)}{\Gamma(\alpha-\beta+1)(-\alpha+\beta-\gamma)}\sum_{n=0}^{\infty}\left(\frac{(\alpha+1)_n(\alpha-\beta+1)_n}{(\alpha-\beta+2)_n n!(\alpha-\beta+1)}-\frac{(\alpha+1)_n(-\gamma+1)_n}{(-\gamma+2)_n n!(-\gamma+1)}\right).
\end{aligned}
$$

Here, if $\Re(\alpha)<0$, then the infinite sums

$$\sum_{n=0}^{\infty}\frac{(\alpha+1)_n(\alpha-\beta+1)_n}{(\alpha-\beta+2)_n n!}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{(\alpha+1)_n(-\gamma+1)_n}{(-\gamma+2)_n n!}$$

converge to the hypergeometric series

$$ {}_2F_1\left[\begin{matrix}\alpha+1,\alpha-\beta+1\\ \alpha-\beta+2\end{matrix};1\right] \quad\text{and}\quad {}_2F_1\left[\begin{matrix}\alpha+1,-\gamma+1\\ -\gamma+2\end{matrix};1\right] $$

which are respectively equal to

$$ \frac{\Gamma(\alpha-\beta+2)\Gamma(-\alpha)}{\Gamma(-\beta+1)} \quad\text{and}\quad \frac{\Gamma(-\gamma+2)\Gamma(-\alpha)}{\Gamma(-\alpha-\gamma+1)} $$

by Gauss's summation theorem. Thus

$$ \begin{aligned} &(-\alpha+\beta-\gamma)\int_{0<t_a<t_b<1}\left(\frac{1}{1-t_a}\right)^{\alpha}\left(\frac{t_b}{t_a}\right)^{\gamma}\left(\frac{1-t_a}{1-t_b}\right)^{\beta}\left(\frac{1}{t_b}\right)^{-\alpha}\frac{dt_a dt_b}{(1-t_a)t_b} \\ &=\frac{\Gamma(\alpha+1)\Gamma(-\beta+1)}{\Gamma(\alpha-\beta+1)}\left(\frac{\Gamma(\alpha-\beta+2)\Gamma(-\alpha)}{\Gamma(-\beta+1)(\alpha-\beta+1)}-\frac{\Gamma(-\gamma+2)\Gamma(-\alpha)}{\Gamma(-\alpha-\gamma+1)(-\gamma+1)}\right) \\ &=\frac{\pi}{\sin(\pi\alpha)}\left(\frac{\Gamma(1-\gamma)\Gamma(1-\beta)}{\Gamma(1-\alpha-\gamma)\Gamma(1+\alpha-\beta)}-1\right) \end{aligned} $$

if $\Re(\alpha) < 0$. By the identity theorem of complex functions, this equality also holds without the condition $\Re(\alpha) < 0$. Thus the proposition is proved. □

Now all propositions in Section 4 are proved, and hence Theorems 3 and 5 are unconditionally proved.

Acknowledgements

This work was supported by JSPS KAKENHI Grant Numbers JP18K03243 and JP18K13392.

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