WEIGHTED SUM FORMULA FOR MULTIPLE HARMONIC SUMS MODULO PRIMES

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Abstract. In this paper we prove a weighted sum formula for multiple harmonic sums modulo primes, thereby proving a weighted sum formula for finite multiple zeta values. Our proof utilizes difference equations for the generating series of multiple harmonic sums. We also conjecture several weighted sum formulas of similar flavor for finite multiple zeta values.

1. Introduction

For integers $k_1, \dots, k_d \geq 1$ and a prime p, we put

$$H_p(k_1, \dots, k_d) := \sum_{0<n_1<\cdots<n_d<p} n_1^{-k_1} \cdots n_d^{-k_d} \in \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z},$$
$$H_p^\star(k_1, \dots, k_d) := \sum_{0<n_1\leq\cdots\leq n_d<p} n_1^{-k_1} \cdots n_d^{-k_d} \in \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}.$$

The purpose of this paper is to prove the following theorem:

Theorem 1 (Weighted sum formula for multiple harmonic sums modulo primes)**.** *Let k be a positive integer, d a positive odd integer, p an odd prime larger than d, and i an integer with $1 \leq i \leq d$. Then we have*

$$\sum_{\substack{k_1,\dots,k_d\geq 1\\ k_1+\cdots+k_d=k}} 2^{k_i} H_p(k_1, \dots, k_d) = \begin{cases} 0 & p-1 \nmid k, \\ -1 & p-1 \mid k. \end{cases}$$

We put $\mathcal{A} := \prod_p \mathbb{F}_p / \bigoplus_p \mathbb{F}_p$. For $k_1, \dots, k_d \in \mathbb{Z}_{\geq 1}$, Kaneko and Zagier [4] defined $\mathcal{A}$-finite multiple zeta(-star) values by

$$\zeta_{\mathcal{A}}(k_1, \dots, k_d) := (H_p(k_1, \dots, k_d))_p \in \mathcal{A},$$
$$\zeta_{\mathcal{A}}^\star(k_1, \dots, k_d) := (H_p^\star(k_1, \dots, k_d))_p \in \mathcal{A}.$$

These are analogues of classical real-valued multiple zeta(-star) values. Theorem 1 implies the following corollary:

Theorem 2 (Weighted sum formula for finite multiple zeta(-star) values)**.** *Let k be a positive integer, d a positive odd integer, and i an integer with $1 \leq i \leq d$. Then we have*

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$$\sum_{\substack{k_1,\dots,k_d\geq 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta_{\mathcal{A}}(k_1,\dots,k_d)=0, \tag{1.1}$$

$$\sum_{\substack{k_1,\dots,k_d\geq 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_d)=0. \tag{1.2}$$

Note that several similar weighted sum formulas for multiple zeta values are already known [1, 5, 6], and somewhat different weighted sum formulas for $\mathcal{A}$-finite multiple zeta values were proved by Kamano in [3].

In Section 2, we prove Theorems 1 and 2. In Section 3, we conjecture several weighted sum formulas of similar flavor.

2. Proof of main theorem

Throughout this section, we fix an arbitrary odd prime p.

2.1. **Restatement of the main theorem by a generating function.** For $x_1,\dots,x_d\in\mathbb{F}_p(x)\setminus\mathbb{F}_p$, we put

$$f(x_1,\dots,x_d):=\sum_{0<n_1<\cdots<n_d<p}\frac{1}{(n_1-x_1)\cdots(n_d-x_d)}\in\mathbb{F}_p(x).$$

Here we understand that $f(x_1,\dots,x_d)=1$ if $d=0$. Then since $1/(n-x)=\sum_{k=1}^{\infty}x^{k-1}n^{-k}$, the series expansion of $f(a_1x,\dots,a_dx)$ at $x=0$ is given by

$$\sum_{k_1,\dots,k_d\geq 1} a_1^{k_1-1}\cdots a_d^{k_d-1}H_p(k_1,\dots,k_d)x^{k_1+\cdots+k_d-d}\in\mathbb{F}_p[[x]]$$

for $a_1,\dots,a_d\in\mathbb{F}_p^{\times}$. Therefore,

$$\sum_{\substack{k_1,\dots,k_d\geq 1\\ k_1+\cdots+k_d=k}} 2^{k_i}H_p(k_1,\dots,k_d)$$

is equal to the coefficient of x^k in

$$2x^d f(\overbrace{x,\dots,x}^{i-1},2x,\overbrace{x,\dots,x}^{d-i})$$

for every $k\geq 0$. Thus Theorem 1 is equivalent to the statement that

$$f(\overbrace{x,\dots,x}^{i-1},2x,\overbrace{x,\dots,x}^{d-i})=\frac{x^{p-1-d}}{2(x^{p-1}-1)}$$

holds for every positive odd integer $d<p$ and $i\in\{1,\dots,d\}$.

2.2. **Evaluation of $f(x,\dots,x)$.**

Lemma 3. *For $0<d<p$, we have*

$$f(\overbrace{x,\dots,x}^{d})=\frac{x^{p-1-d}}{x^{p-1}-1}.$$

Proof. Put $a_0 = 1$ and $a_d = f(\overbrace{x, \ldots, x}^{d})$ for $0 < d < p$. Then we have

$$\prod_{0<n<p} \left(1 + \frac{y}{n - x}\right) = \sum_{d=0}^{p-1} a_d y^d.$$

On the other hand,

$$\begin{aligned}\prod_{0<n<p} \left(1 + \frac{y}{n - x}\right) &= \prod_{0<n<p} \frac{n - x + y}{n - x} \\ &= \frac{(x - y)^{p-1} - 1}{x^{p-1} - 1} \\ &= 1 + \sum_{d=1}^{p-1} \frac{x^{p-1-d}}{x^{p-1} - 1} y^d\end{aligned}$$

in $\mathbb{F}_p(x)[y]$. Thus $a_d = x^{p-1-d}/(x^{p-1} - 1)$ for $0 < d < p$. □

2.3. **Difference equations.** Put $\tilde{f}(x_0, x_1, \ldots, x_d, x_{d+1}) := \frac{1}{x_0 x_{d+1}} f(x_1, \ldots, x_d)$ for $x_0, \ldots, x_{d+1} \in \mathbb{F}_p(x) \setminus \mathbb{F}_p$. For $a, b \in \mathbb{Z}_{\geq -1}$ with $(a, b) \neq (-1, -1)$, we put

$$\begin{aligned}G_{a,b}(x) :=& x^2 \tilde{f}(\overbrace{x, \ldots, x}^{a+1}, 2x, \overbrace{x, \ldots, x}^{b+1}) \\ =& \begin{cases} f(\overbrace{x, \ldots, x}^{a}, 2x, \overbrace{x, \ldots, x}^{b}) & a \geq 0 \text{ and } b \geq 0, \\ \frac{1}{2} f(\overbrace{x, \ldots, x}^{a+b+1}) & a = -1 \text{ or } b = -1 \end{cases}\end{aligned}$$

and

$$\begin{aligned}S_{a,b}(x) :=& x^2 \tilde{f}(\overbrace{x, \ldots, x}^{a+1}, 2x - 1, \overbrace{x, \ldots, x}^{b+1}) \\ =& \begin{cases} f(\overbrace{x, \ldots, x}^{a}, 2x - 1, \overbrace{x, \ldots, x}^{b}) & a \geq 0 \text{ and } b \geq 0, \\ \frac{x}{2x-1} f(\overbrace{x, \ldots, x}^{a+b+1}) & a = -1 \text{ or } b = -1. \end{cases}\end{aligned}$$

Lemma 4. *For $d \geq 1$, we have*

$$\begin{aligned}&f(x_1 - 1, \ldots, x_d - 1) - f(x_1, \ldots, x_d) \\ &\qquad = \frac{1}{x_1 - 1} f(x_2 - 1, \ldots, x_d - 1) - \frac{1}{x_d} f(x_1, \ldots x_{d-1}).\end{aligned}$$

Proof. Since

$$\begin{aligned}&\{1 < n_1 < \cdots < n_d \leq p\} \sqcup \{1 = n_1 < \cdots < n_d \leq p\} \\ &\qquad = \{0 < n_1 < \cdots < n_d < p\} \sqcup \{0 < n_1 < \cdots < n_d = p\},\end{aligned}$$

we have

$$\begin{aligned}
&f(x_1-1,\dots,x_d-1)\\
=&\sum_{1<n_1<\cdots<n_d\le p}\frac{1}{(n_1-x_1)\cdots(n_d-x_d)}\\
=&\sum_{0<n_1<\cdots<n_d<p}\frac{1}{(n_1-x_1)\cdots(n_d-x_d)}+\sum_{0<n_1<\cdots<n_d=p}\frac{1}{(n_1-x_1)\cdots(n_d-x_d)}\\
&-\sum_{1=n_1<\cdots<n_d\le p}\frac{1}{(n_1-x_1)\cdots(n_d-x_d)}\\
=&f(x_1,\dots,x_d)-\frac{1}{x_d}f(x_1,\dots x_{d-1})-\frac{1}{1-x_1}f(x_2-1,\dots,x_d-1).
\end{aligned}$$

□

Lemma 5. *For $a,b\ge 0$, we have*

$$G_{a,b}(x-1)-S_{a,b}(x)=\frac{1}{x-1}G_{a-1,b}(x-1)-\frac{1}{x}S_{a,b-1}(x).$$

Proof. Applying Lemma 4 to the case $d=a+b+1$ and $(x_1,\dots,x_d)=(\overbrace{x,\dots,x}^{a},2x-1,\overbrace{x,\dots,x}^{b})$, we have

$$\begin{aligned}
&G_{a,b}(x-1)-S_{a,b}(x)\\
&=f(x_1-1,\dots,x_d-1)-f(x_1,\dots,x_d)\\
&=\frac{1}{x_1-1}f(x_2-1,\dots,x_d-1)-\frac{1}{x_d}f(x_1,\dots,x_{d-1})\\
&=(x-1)\tilde f(x_1-1,\dots,x_d-1,x-1)-x\tilde f(x,x_1,\dots,x_d)\\
&=\frac{1}{x-1}G_{a-1,b}(x-1)-\frac{1}{x}S_{a,b-1}(x).
\end{aligned}$$

□

Lemma 6. *For $1\le i\le d$, we have*

$$\begin{aligned}
&\tilde f(x_0,\dots,x_{i-1},x_i-1,x_{i+1},\dots,x_{d+1})-\tilde f(x_0,\dots,x_{d+1})\\
=&\frac{1}{x_i-x_{i-1}-1}\left(\tilde f(x_0,\dots,x_{i-1},x_{i+1},\dots,x_{d+1})-\tilde f(x_0,\dots,x_{i-2},x_i-1,x_{i+1},\dots,x_{d+1})\right)\\
&+\frac{1}{x_i-x_{i+1}}\left(\tilde f(x_0,\dots,x_i,x_{i+2},\dots,x_{d+1})-\tilde f(x_0,\dots,x_{i-1},x_{i+1},\dots,x_{d+1})\right).
\end{aligned}$$

Proof. Suppose that $1\le i\le d$. We define four subsets S_1,S_2,S_3,S_4 of $\{(n_0,\dots,n_{d+1})\in\mathbb{Z}^{d+2}\mid n_0=0,\ n_{d+1}=p\}$ by

$$\begin{aligned}
S_1:=&\{0=n_0<\cdots<n_{i-1}<n_i-1<n_{i+1}<\cdots<n_{d+1}=p\},\\
S_2:=&\{0=n_0<\cdots<n_{i-1}=n_i-1<n_{i+1}<\cdots<n_{d+1}=p\},\\
S_3:=&\{0=n_0<\cdots<n_{i-1}<n_i<n_{i+1}<\cdots<n_{d+1}=p\},\\
S_4:=&\{0=n_0<\cdots<n_{i-1}<n_i=n_{i+1}<\cdots<n_{d+1}=p\}.
\end{aligned}$$

Put

$$A_k:=\sum_{(n_0,\dots,n_{d+1})\in S_k}\prod_{j=0}^{d+1}\frac{1}{n_j-x_j}$$

for $k \in \{1,2,3,4\}$. It is immediate from the definition that

$$\begin{aligned} A_1 &= \tilde{f}(x_0,\dots,x_{i-1},x_i-1,x_{i+1},\dots,x_{d+1}),\\ A_3 &= \tilde{f}(x_0,\dots,x_{d+1}). \end{aligned}$$

We have

$$\begin{aligned} A_2 =& \sum_{(n_0,\dots,n_{d+1})\in S_2} \frac{1}{(n_{i-1}-x_{i-1})(n_i-x_i)} \prod_{j\notin\{i-1,i\}} \frac{1}{n_j-x_j}\\ =&\frac{1}{1-x_i+x_{i-1}} \sum_{(n_0,\dots,n_{d+1})\in S_2} \left(\frac{1}{n_{i-1}-x_{i-1}} - \frac{1}{n_i-x_i}\right) \prod_{j\notin\{i-1,i\}} \frac{1}{n_j-x_j}\\ =&\frac{1}{1-x_i+x_{i-1}} \left(\tilde{f}(x_0,\dots,x_{i-1},x_{i+1},\dots,x_{d+1}) - \tilde{f}(x_0,\dots,x_{i-2},x_i-1,x_{i+1},\dots,x_{d+1})\right) \end{aligned}$$

and

$$\begin{aligned} A_4 &= \sum_{(n_0,\dots,n_{d+1})\in S_4} \frac{1}{(n_i-x_i)(n_{i+1}-x_{i+1})} \prod_{j\notin\{i,i+1\}} \frac{1}{n_j-x_j}\\ &= \frac{1}{x_i-x_{i+1}} \sum_{(n_0,\dots,n_{d+1})\in S_4} \left(\frac{1}{n_i-x_i} - \frac{1}{n_{i+1}-x_{i+1}}\right) \prod_{j\notin\{i,i+1\}} \frac{1}{n_j-x_j}\\ &= \frac{1}{x_i-x_{i+1}} \left(\tilde{f}(x_0,\dots,x_i,x_{i+2},\dots,x_{d+1}) - \tilde{f}(x_0,\dots,x_{i-1},x_{i+1},\dots,x_{d+1})\right). \end{aligned}$$

Since $S_1 \sqcup S_2 = S_3 \sqcup S_4$, we have $A_1 - A_3 = -A_2 + A_4$. Thus

$$\begin{aligned} &\tilde{f}(x_0,\dots,x_{i-1},x_i-1,x_{i+1},\dots,x_{d+1}) - \tilde{f}(x_0,\dots,x_{d+1})\\ =&\frac{1}{x_i-x_{i-1}-1} \left(\tilde{f}(x_0,\dots,x_{i-1},x_{i+1},\dots,x_{d+1}) - \tilde{f}(x_0,\dots,x_{i-2},x_i-1,x_{i+1},\dots,x_{d+1})\right)\\ &+ \frac{1}{x_i-x_{i+1}} \left(\tilde{f}(x_0,\dots,x_i,x_{i+2},\dots,x_{d+1}) - \tilde{f}(x_0,\dots,x_{i-1},x_{i+1},\dots,x_{d+1})\right). \qquad \square \end{aligned}$$

Lemma 7. *For $a,b \geq 0$, we have*

$$S_{a,b}(x) - G_{a,b}(x) = \frac{1}{x(x-1)} f(\overbrace{x,\dots x}^{a+b}) - \frac{1}{x-1} S_{a-1,b}(x) + \frac{1}{x} G_{a,b-1}(x).$$

Proof. Let us apply Lemma 6 to the case $d = a+b+1$, $i = a+1$ and $(x_0, \dots, x_{d+1}) = (\overbrace{x, \dots, x}^{a+1}, 2x, \overbrace{x, \dots, x}^{b+1})$. Then since

$$\begin{aligned}
\tilde{f}(x_0, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_{d+1}) &= \tilde{f}(\overbrace{x, \dots, x}^{a+1}, 2x-1, \overbrace{x, \dots, x}^{b+1}) \\
&= \frac{1}{x^2} S_{a,b}(x), \\
\tilde{f}(x_0, \dots, x_{d+1}) &= \tilde{f}(\overbrace{x, \dots, x}^{a+1}, 2x, \overbrace{x, \dots, x}^{b+1}) \\
&= \frac{1}{x^2} G_{a,b}(x), \\
\tilde{f}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1}) &= \tilde{f}(\overbrace{x, \dots, x}^{a+b+2}) \\
&= \frac{1}{x^2} f(\overbrace{x, \dots, x}^{a+b}), \\
\tilde{f}(x_0, \dots, x_{i-2}, x_i - 1, x_{i+1}, \dots, x_{d+1}) &= \tilde{f}(\overbrace{x, \dots, x}^{a}, 2x-1, \overbrace{x, \dots, x}^{b+1}) \\
&= \frac{1}{x^2} S_{a-1,b}(x), \\
\tilde{f}(x_0, \dots, x_i, x_{i+2}, \dots, x_{d+1}) &= \tilde{f}(\overbrace{x, \dots, x}^{a+1}, 2x, \overbrace{x, \dots, x}^{b}) \\
&= \frac{1}{x^2} G_{a,b-1}(x),
\end{aligned}$$

we have

$$\begin{aligned}
\frac{1}{x^2} S_{a,b}(x) - \frac{1}{x^2} G_{a,b}(x) = \frac{1}{x-1} &\left(\frac{1}{x^2} f(\overbrace{x, \dots, x}^{a+b}) - \frac{1}{x^2} S_{a-1,b}(x) \right) \\
&+ \frac{1}{x} \left(\frac{1}{x^2} G_{a,b-1}(x) - \frac{1}{x^2} f(\overbrace{x, \dots, x}^{a+b}) \right).
\end{aligned} \tag{2.1}$$

By multiplying x^2 to both sides of Equation (2.1), we obtain the lemma. □

2.4. **Proof of the main theorem.** Put $U(x) := x^{p-1}/(x^{p-1} - 1)$. Note that $f(\overbrace{x, \dots, x}^{d}) = x^{-d} U(x)$ for $0 < d < p$ by Lemma 3. By the argument in Section 2.1, Theorem 1 is equivalent to saying that

$$2G_{a,b}(x) = x^{-a-b-1} U(x) \tag{2.2}$$

for $a, b \geq 0$ such that $a+b$ is an even integer less than $p-1$. We will prove (2.2) for $a, b \geq -1$ such that $a+b$ is a non-negative even integer less than $p-1$ by induction on $a+b$. First, (2.2) is obvious if $a = -1$, $b = -1$, or $(a, b) = (0, 0)$.

Assume that $a \geq 0$, $b \geq 0$, and $a+b \geq 2$. By Lemmas 5 and 7, we have

$$\begin{aligned} G_{a,b}(x-1) - G_{a,b}(x) = &-\frac{1}{x}S_{a,b-1}(x) + \frac{1}{x}G_{a,b-1}(x) \\ &+ \frac{1}{x-1}G_{a-1,b}(x-1) - \frac{1}{x-1}S_{a-1,b}(x) \\ &+ \frac{U(x)}{x^{a+b+1}(x-1)}. \end{aligned} \tag{2.3}$$

By Lemma 7 and the induction hypothesis, for $b > 0$ we have

$$\begin{aligned} -\frac{1}{x}S_{a,b-1}(x) + \frac{1}{x}G_{a,b-1}(x) = &-\frac{f(\overbrace{x,\dots x}^{a+b-1})}{x^2(x-1)} + \frac{1}{x(x-1)}S_{a-1,b-1}(x) - \frac{1}{x^2}G_{a,b-2}(x) \\ =&\frac{U(x)}{x^{a+b+1}}\left(-\frac{1}{x-1} - \frac{1}{2}\right) + \frac{1}{x(x-1)}S_{a-1,b-1}(x). \end{aligned} \tag{2.4}$$

We can check by direct computation that (2.4) is also true for $b = 0$.

By Lemma 5 and the induction hypothesis, for $a > 0$ we have

$$\begin{aligned} \frac{1}{x-1}G_{a-1,b}(x-1) - \frac{1}{x-1}S_{a-1,b}(x) =&\frac{1}{(x-1)^2}G_{a-2,b}(x-1) - \frac{1}{x(x-1)}S_{a-1,b-1}(x) \\ =&\frac{U(x-1)}{2(x-1)^{a+b+1}} - \frac{1}{x(x-1)}S_{a-1,b-1}(x). \end{aligned} \tag{2.5}$$

We can check by direct computation that (2.5) is also true for $a = 0$.

By (2.3), (2.4) and (2.5),

$$G_{a,b}(x-1) - \frac{U(x-1)}{2(x-1)^{a+b+1}} = G_{a,b}(x) - \frac{U(x)}{2x^{a+b+1}}.$$

Put $q(x) := G_{a,b}(x) - \frac{U(x)}{2x^{a+b+1}}$. Let

$$q(x) = \sum_{n=0}^{p-1}\sum_{k=0}^{\infty}\frac{c_{n,k}}{(x-n)^k}$$

be a partial fractional decomposition. Since $q(x-1) = q(x)$, we have $c_{0,k} = c_{1,k} = \cdots = c_{p-1,k}$. Since $a+b < p-1$, we see that $q(x)$ has no pole at $x = 0$, which implies that $c_{0,k} = 0$. Thus $q(x) = 0$, which proves (2.2).

Now we prove Theorem 2. The equality (1.1) is an immediate consequence of Theorem 1. To prove (1.2), we use the following well-known lemma (for the proof of this lemma, see [7, Prop 2.9] for example).

Lemma 8. *For $d \geq 1$ and $k_1, \dots, k_d \in \mathbb{Z}_{\geq 1}$, we have*

$$\sum_{j=0}^{d}(-1)^j \zeta_{\mathcal{A}}^{\star}(k_1,\dots,k_j)\zeta_{\mathcal{A}}(k_d,\dots,k_{j+1}) = 0.$$

Here, we understand that $\zeta_{\mathcal{A}}^{\star}(k_1,\dots,k_j) = \zeta_{\mathcal{A}}^{\star}(\emptyset) = 1$ if $j = 0$, and that $\zeta_{\mathcal{A}}(k_d,\dots,k_{j+1}) = \zeta_{\mathcal{A}}(\emptyset) = 1$ if $j = d$.

It is well known that

$$\sum_{\substack{k_1,\dots,k_s\ge 1\\ k_1+\cdots+k_s=k}} \zeta_{\mathcal{A}}(k_1,\dots,k_s)=0, \tag{2.6}$$

$$\sum_{\substack{k_1,\dots,k_s\ge 1\\ k_1+\cdots+k_s=k}} \zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_s)=0 \tag{2.7}$$

for $s\ge 1$ (see [2]). Recall that we need to prove

$$\sum_{\substack{k_1,\dots,k_d\ge 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_d)=0$$

for a positive integer k, a positive odd integer d, and an integer i with $1\le i\le d$. By considering the sum $\sum_{k_1,\dots,k_d} 2^{k_i}(\cdots)$ of Lemma 8, we have

$$\sum_{j=0}^{d}(-1)^j \sum_{\substack{k_1,\dots,k_d\ge 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_j)\zeta_{\mathcal{A}}(k_d,\dots,k_{j+1})=0. \tag{2.8}$$

By (2.6), we have

$$\sum_{\substack{k_1,\dots,k_d\ge 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_j)\zeta_{\mathcal{A}}(k_d,\dots,k_{j+1})=0 \quad (i\le j\le d-1) \tag{2.9}$$

since

$$\begin{aligned}
&\sum_{\substack{k_1,\dots,k_d\ge 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_j)\zeta_{\mathcal{A}}(k_d,\dots,k_{j+1})\\
&=\sum_{l+m=k}\left(\sum_{\substack{k_1,\dots,k_j\ge 1\\ k_1+\cdots+k_j=l}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_j)\right)\left(\sum_{\substack{k_{j+1},\dots,k_d\ge 1\\ k_{j+1}+\cdots+k_d=m}} \zeta_{\mathcal{A}}(k_d,\dots,k_{j+1})\right)\\
&=\sum_{l+m=k}\left(\sum_{\substack{k_1,\dots,k_j\ge 1\\ k_1+\cdots+k_j=l}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_j)\right)\cdot 0=0.
\end{aligned}$$

Similarly, by (2.7), we have

$$\sum_{\substack{k_1,\dots,k_d\ge 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_j)\zeta_{\mathcal{A}}(k_d,\dots,k_{j+1})=0 \quad (1\le j< i). \tag{2.10}$$

By (2.8), (2.9), and (2.10), we have

$$\sum_{\substack{k_1,\dots,k_d\ge 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta_{\mathcal{A}}(k_d,\dots,k_1)=(-1)^{d+1}\sum_{\substack{k_1,\dots,k_d\ge 1\\ k_1+\cdots+k_d=k}} 2^{k_i}\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_d).$$

Therefore (1.2) follows from (1.1). Thus Theorem 2 is proved.

3. Further conjectures

We define a general weighted sum of finite multiple zeta values by

$$W_k(n_1,\dots,n_d) = \sum_{\substack{k_1,\dots,k_d\geq 1\\ k_1+\cdots+k_d=k}} n_1^{k_1}\cdots n_d^{k_d}\zeta_{\mathcal{A}}(k_1,\dots,k_d),$$

$$W_k^{\star}(n_1,\dots,n_d) = \sum_{\substack{k_1,\dots,k_d\geq 1\\ k_1+\cdots+k_d=k}} n_1^{k_1}\cdots n_d^{k_d}\zeta_{\mathcal{A}}^{\star}(k_1,\dots,k_d).$$

We list several conjectural weighted sum formulas for finite multiple zeta values that are different from Theorem 2.

Conjecture 9. *For $k, r \in \mathbb{Z}_{\geq 1}$, we have*

$$W_k(1,1,2,3,\dots,r-1,r) \stackrel{?}{=} 0,$$

$$W_k^{\star}(1,1,2,3,\dots,r-1,r) \stackrel{?}{=} 0.$$

Conjecture 10. *For $k \in \mathbb{Z}_{\geq 1}$ and a positive odd integer r, we have*

$$W_k(1,1,2,3,\dots,r-2,r-1,r,r) \stackrel{?}{=} 0,$$

$$W_k^{\star}(1,1,2,3,\dots,r-2,r-1,r,r) \stackrel{?}{=} 0.$$

Conjecture 11. *For $k, r \in \mathbb{Z}_{\geq 1}$ and $a, b \in \mathbb{Q}$, we have*

$$W_k(a,a+b,a+2b,\dots,a+rb) \stackrel{?}{=} W_k(b,a+b,a+2b,\dots,a+rb),$$

$$W_k^{\star}(a,a+b,a+2b,\dots,a+rb) \stackrel{?}{=} W_k^{\star}(b,a+b,a+2b,\dots,a+rb).$$

Remark 12. Conjecture 9 is a special case of Conjecture 11.

Remark 13. We checked the p-components of Conjecture 9 for $1 \leq k \leq 10$, $1 \leq r \leq 10$, and $p \in \{31, 37, 41, 43, 47\}$. We checked the p-components of Conjecture 10 for $1 \leq k \leq 10$, $1 \leq r \leq 9$, and $p \in \{31, 37, 41, 43, 47\}$. We checked the p-components of Conjecture 11 for $1 \leq k \leq 10$, $1 \leq r \leq 5$, $a, b \in \{0, \pm 1, \pm\frac{1}{2}, 2, 5\}$, and $p \in \{31, 37, 41, 43, 47\}$.

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