

ANALOGUES OF THE AOKI-OHNO AND LE-MURAKAMI RELATIONS FOR FINITE MULTIPLE ZETA VALUES

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Abstract

We establish finite analogues of the identities known as the Aoki-Ohno relation and the Le-Murakami relation in the theory of multiple zeta values. We use an explicit form of a generating series given by Aoki and Ohno.

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1. Introduction and the statement

For an index set of positive integers $\mathbf{k} = (k_1, \dots, k_r)$ with $k_1 > 1$, the multiple zeta value $\zeta(\mathbf{k})$ and the multiple zeta-star value $\zeta^*(\mathbf{k})$ are defined respectively by the nested series

$$\zeta(\mathbf{k}) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}$$

and

$$\zeta^*(\mathbf{k}) = \sum_{m_1 \geq \dots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}.$$

We refer to the sum $k_1 + \dots + k_r$, the length r , and the number of components k_i with $k_i > 1$ as the weight, depth, and height of the index \mathbf{k} respectively.

For given k and s , let $I_0(k, s)$ be the set of indices $\mathbf{k} = (k_1, \dots, k_r)$ with $k_1 > 1$ of weight k and height s . We naturally have $k \geq 2s$ and $s \geq 1$; otherwise $I_0(k, s)$ is empty.

Aoki and Ohno proved in [1] the identity

$$\sum_{\mathbf{k} \in I_0(k, s)} \zeta^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) \zeta(k). \quad (1.1)$$

On the other hand, for $\zeta(\mathbf{k})$, the following identity is known as the Le-Murakami relation ([6]): For even k , it holds

$$\sum_{\mathbf{k} \in I_0(k,s)} (-1)^{\text{dep}(\mathbf{k})} \zeta(\mathbf{k}) = \frac{(-1)^{k/2}}{(k+1)!} \sum_{r=0}^{k/2-s} \binom{k+1}{2r} (2-2^{2r}) B_{2r} \pi^k,$$

where B_n denotes the Bernoulli number. By Euler, the right-hand side is a rational multiple of the Riemann zeta value $\zeta(k)$.

In this short article, we establish their analogous identities for *finite multiple zeta values*.

For an index set of positive integers $\mathbf{k} = (k_1, \dots, k_r)$, the finite multiple zeta value $\zeta_{\mathcal{A}}(\mathbf{k})$ and the finite multiple zeta-star value $\zeta_{\mathcal{A}}^*(\mathbf{k})$ are elements in the quotient ring $\mathcal{A} := (\prod_p \mathbb{Z}/p\mathbb{Z}) / (\bigoplus_p \mathbb{Z}/p\mathbb{Z})$ (p runs over all primes) represented respectively by

$$\left(\sum_{p > m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \text{ mod } p \right)_p \quad \text{and} \quad \left(\sum_{p > m_1 \geq \dots \geq m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \text{ mod } p \right)_p.$$

Studies of finite multiple zeta(-star) values go back at least to Hoffman [2] (its preprint was available around 2004) and Zhao [10]. But it was rather recent that Zagier proposed (in 2012 to the first-named author) considering them in the (characteristic 0) ring \mathcal{A} ([5], see [3, 4]). In \mathcal{A} , the naive analogue $\zeta_{\mathcal{A}}(k)$ of the Riemann zeta value $\zeta(k)$ is zero because $\sum_{n=1}^{p-1} 1/n^k$ is congruent to 0 modulo p for all sufficiently large primes p . However, the "true" analogue of $\zeta(k)$ in \mathcal{A} is considered to be

$$Z(k) := \left(\frac{B_{p-k}}{k} \right)_p.$$

We note that this value is zero when k is even because the odd-indexed Bernoulli numbers are 0 except B_1 . It is still an open problem whether $Z(k) \neq 0$ for any odd $k \geq 3$.

Now we state our main theorem, where the role of $Z(k)$ as a finite analogue of $\zeta(k)$ is evident.

THEOREM 1.1. *The following identities hold in \mathcal{A} :*

$$\sum_{\mathbf{k} \in I_0(k,s)} \zeta_{\mathcal{A}}^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1-2^{1-k}) Z(k), \quad (1.2)$$

$$\sum_{\mathbf{k} \in I_0(k,s)} (-1)^{\text{dep}(\mathbf{k})} \zeta_{\mathcal{A}}(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1-2^{1-k}) Z(k). \quad (1.3)$$

We should note that the right-hand sides are exactly the same. In the next section, we give a proof of the theorem.

2. Proof

Let $\text{Li}_{\mathbf{k}}^*(t)$ be the “non-strict” version of the multiple-polylogarithm:

$$\text{Li}_{\mathbf{k}}^*(t) = \sum_{m_1 \geq \dots \geq m_r \geq 1} \frac{t^{m_1}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

Aoki and Ohno [1] computed the generating function

$$\Phi_0 := \sum_{k, s \geq 1} \left(\sum_{\mathbf{k} \in I_0(k, s)} \text{Li}_{\mathbf{k}}^*(t) \right) x^{k-2s} z^{2s-2},$$

and, in view of $\text{Li}_{\mathbf{k}}^*(1) = \zeta^*(\mathbf{k})$ (if $k_1 > 1$), evaluated it at $t = 1$ to obtain the identity (1.1). For our purpose, the function $\text{Li}_{\mathbf{k}}^*(t)$ is useful because the truncated sum

$$\sum_{p > m_1 \geq \dots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

used to define $\zeta_{\mathcal{A}}^*(\mathbf{k})$ is the sum of the coefficients of t^i in $\text{Li}_{\mathbf{k}}^*(t)$ for $i = 1, \dots, p-1$. In §3 of [1], they showed

$$\Phi_0 = \sum_{n=1}^{\infty} a_n t^n,$$

where

$$a_n = \sum_{l=1}^n \left(\frac{A_{n,l}(z)}{x+z-l} + \frac{A_{n,l}(-z)}{x-z-l} \right)$$

and

$$A_{n,l}(z) = (-1)^l \binom{n-1}{l-1} \frac{(z-l+1) \cdots (z-1) z (z+1) \cdots (z+n-l-1)}{(2z-l+1) \cdots (2z-1) 2z (2z+1) \cdots (2z+n-l)}.$$

The problem is then to compute the coefficient of $x^{k-2s} z^{2s-2}$ in $\sum_{n=1}^{p-1} a_n$ modulo p .

We proceed as follows:

$$\begin{aligned} \sum_{n=1}^{p-1} a_n &= \sum_{n=1}^{p-1} \sum_{l=1}^n \left(\frac{A_{n,l}(z)}{x+z-l} + \frac{A_{n,l}(-z)}{x-z-l} \right) \\ &= \sum_{l=1}^{p-1} \sum_{n=l}^{p-1} \left(\frac{A_{n,l}(z)}{x+z-l} + \frac{A_{n,l}(-z)}{x-z-l} \right) \\ &= \sum_{l=1}^{p-1} \sum_{n=0}^{p-l-1} \left(\frac{A_{n+l,l}(z)}{x+z-l} + \frac{A_{n+l,l}(-z)}{x-z-l} \right). \end{aligned}$$

Writing $A_{n+l,l}(z)$ as

$$A_{n+l,l}(z) = \frac{(-1)^l}{2z} \frac{(z-l+1)_{l-1}}{(2z-l+1)_{l-1}} \frac{(l)_n(z)_n}{(2z+1)_n n!},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, we have

$$\sum_{n=0}^{p-l-1} A_{n+l,l}(z) = \frac{(-1)^l}{2z} \frac{(z-l+1)_{l-1}}{(2z-l+1)_{l-1}} \sum_{n=0}^{p-l-1} \frac{(l)_n(z)_n}{(2z+1)_n n!}.$$

We view the sum on the right as

$$\sum_{n=0}^{p-l-1} \frac{(l)_n(z)_n}{(2z+1)_n n!} \equiv F(-p+l, z; 2z+1; 1) - \frac{(l)_{p-l}(z)_{p-l}}{(2z+1)_{p-l}(p-l)!} \pmod{p}.$$

Here, $F(a, b; c; z)$ is the Gauss hypergeometric series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $(a)_n$ for $n \geq 1$ is as before and $(a)_0 = 1$. Note that if a (or b) is a non-positive integer $-m$, then $F(a, b; c; z)$ is a polynomial in z of degree at most m , and the renowned formula of Gauss

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

becomes

$$F(-m, b; c; 1) = \frac{(c-b)_m}{(c)_m}.$$

Hence

$$F(-p+l, z; 2z+1; 1) = \frac{(z+1)_{p-l}}{(2z+1)_{p-l}} \equiv \frac{z^{p-1}-1}{(2z)^{p-1}-1} \frac{(2z-l+1)_{l-1}}{(z-l+1)_{l-1}} \pmod{p}.$$

We also compute

$$\frac{(l)_{p-l}(z)_{p-l}}{(2z+1)_{p-l}(p-l)!} \equiv (-1)^{l-1} \frac{z(z^{p-1}-1)}{(2z)^{p-1}-1} \frac{(2z-l+1)_{l-1}}{(z-l)_l} \pmod{p}.$$

Since we only need the coefficient of z^{2s-2} , we may work modulo higher power of z and in particular we may replace $(z^{p-1}-1)/((2z)^{p-1}-1)$ by 1, assuming p is large enough (we may assume this because an identity in \mathcal{A} holds true if the p -components on both sides agree in $\mathbb{Z}/p\mathbb{Z}$ for all large enough p). Hence we get

$$\begin{aligned} \sum_{n=1}^{p-1} a_n &\equiv \sum_{l=1}^{p-1} \left\{ \frac{(-1)^l}{2z} \left(\frac{1}{x+z-l} - \frac{1}{x-z-l} \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{1}{(x+z-l)(z-l)} - \frac{1}{(x-z-l)(z+l)} \right) \right\} \pmod{p}. \end{aligned}$$

By the binomial expansion, we have

$$\begin{aligned} \sum_{l=1}^{p-1} \frac{(-1)^l}{x+z-l} &= \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l} \sum_{m=0}^{\infty} \left(\frac{x+z}{l}\right)^m \\ &= \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l} \sum_{m=0}^{\infty} \frac{1}{l^m} \sum_{i=0}^m \binom{m}{i} x^{m-i} z^i \\ &= \sum_{m \geq l \geq 0} \binom{m}{i} \left(\sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l^{m+1}} \right) x^{m-i} z^i. \end{aligned}$$

From this we obtain

$$\sum_{l=1}^{p-1} \frac{(-1)^l}{2z} \left(\frac{1}{x+z-l} - \frac{1}{x-z-l} \right) = \sum_{m \geq 2i+1 \geq 0} \binom{m}{2i+1} \left(\sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l^{m+1}} \right) x^{m-2i-1} z^{2i}$$

and the coefficient of $x^{k-2s} z^{2s-2}$ in this is (by letting $i \rightarrow s-1$ and $m \rightarrow k-1$)

$$\binom{k-1}{2s-1} \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l^k}.$$

This is known to be congruent modulo p to

$$2 \binom{k-1}{2s-1} (1-2^{1-k}) \frac{B_{p-k}}{k}$$

(see *e.g.* [11, Theorem 8.2.7]). Concerning the other term

$$\begin{aligned} &\sum_{l=1}^{p-1} \frac{1}{2} \left(\frac{1}{(x+z-l)(z-l)} - \frac{1}{(x-z-l)(z+l)} \right) \\ &= \frac{1}{2} \sum_{l=1}^{p-1} \left\{ \frac{1}{x} \left(\frac{1}{z-l} - \frac{1}{x+z-l} \right) - \frac{1}{x} \left(\frac{1}{z+l} + \frac{1}{x-z-l} \right) \right\}, \end{aligned}$$

every quantity that appears as coefficients in the expansion into power series in x and z is a multiple of the sum of the form $\sum_{l=1}^{p-1} 1/l^m$, and are all congruent to 0 modulo p . This concludes the proof of (1.2).

We may prove (1.3) in a similar manner by using the generating series of Ohno-Zagier [7], but we reduce (1.3) to (1.2) by showing that the left-hand sides of both formulas are equal up to sign.

$$\text{Set } S_{k,s} := \sum_{\mathbf{k} \in I_0(k,s)} (-1)^{\text{dep}(\mathbf{k})} \zeta_{\mathcal{A}}^*(\mathbf{k}) \text{ and } S_{k,s}^* := \sum_{\mathbf{k} \in I_0(k,s)} \zeta_{\mathcal{A}}^*(\mathbf{k}).$$

LEMMA 2.1. $S_{k,s}^* = (-1)^{k-1} S_{k,s}$.

PROOF. We use the well-known identity (see for instance [8, Corollary 3.16])

$$\sum_{i=0}^r (-1)^i \zeta_{\mathcal{A}}(k_i, \dots, k_1) \zeta_{\mathcal{A}}^*(k_{i+1}, \dots, k_r) = 0. \quad (2.1)$$

Taking the sum of this over all $\mathbf{k} \in I_0(k, s)$ and separating the terms corresponding to $i = 0$ and $i = r$, we obtain

$$S_{k,s}^* + \sum_{\substack{k'+k''=k \\ s'+s''=s}} \left(\sum_{\mathbf{k}' \in I_0(k', s')} (-1)^{\text{dep}(\mathbf{k}')} \zeta_{\mathcal{A}}(\overleftarrow{\mathbf{k}'}) \right) \left(\sum_{\mathbf{k}'' \in I(k'', s'')} \zeta_{\mathcal{A}}^*(\mathbf{k}'') \right) + (-1)^k S_{k,s} = 0.$$

Here, $\overleftarrow{\mathbf{k}'}$ denotes the reversal of \mathbf{k}' , and the set $I(k'', s'')$ consists of all indices (no restriction on the first component) of weight k'' and height s'' . We have used $\zeta_{\mathcal{A}}(\overleftarrow{\mathbf{k}}) = (-1)^k \zeta_{\mathcal{A}}(\mathbf{k})$ in computing the last term ($i = r$). Since the second sum in the middle is symmetric and hence 0 (by using Hoffman [2, Theorem 4.4] and $\zeta_{\mathcal{A}}(k) = 0$ for all $k \geq 1$), we prove the lemma. \square

Since $Z(k) = 0$ if k is even, we see from Lemma 2.1 that the formula for $S_{k,s}$ is the same as that for $S_{k,s}^*$. This concludes the proof of our theorem.

REMARK 2.2. K. Yaeo [9] proved the lemma in the case $s = 1$ and T. Murakami (unpublished) in general for all odd k .

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References

- [1] T. Aoki and Y. Ohno, Sum relations for multiple zeta values and connection formulas for the Gauss hypergeometric functions, *Publ. RIMS, Kyoto Univ.*, **41** (2005), 329–337.
- [2] M. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, *Kyushu J. Math.*, **69** (2015), 345–366.
- [3] M. Kaneko, Finite multiple zeta values (in Japanese), *RIMS Kôkyûroku Bessatsu*, **B68** (2017), 175–190.
- [4] M. Kaneko, An introduction to classical and finite multiple zeta values, to appear in *Publications Matématiques de Besançon*.
- [5] M. Kaneko and D. Zagier, Finite multiple zeta values, *in preparation*.
- [6] T. Q. T. Le and J. Murakami, Kontsevich’s integral for the Homfly polynomial and relation between values of multiple zeta functions, *Topology Appl.*, **62** (1995), 193–206.
- [7] Y. Ohno and D. Zagier, Multiple zeta values of fixed weight, depth, and height, *Ingag. Math.*, **12** (2001), 483–487.
- [8] K. Sakugawa and S. Seki, On functional equations of finite multiple polylogarithms, *J. Algebra*, **469** (2017), 323–357.
- [9] K. Yaeo, Consideration on relations among finite multiple zeta values (in Japanese), Master’s thesis, Tohoku University, 2017.
- [10] J. Zhao, Wolstenholme type theorem for multiple harmonic sums *Int. J. Number Theory*, **4-1** (2008), 73–106.
- [11] J. Zhao, *Multiple zeta functions, multiple polylogarithms and their special values*, Series on Number Theory and Its Applications, **12**, World Scientific, (2016).

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