SUM FORMULA FOR FINITE MULTIPLE ZETA VALUES

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Abstract. The sum formula is one of the most well-known relations among multiple zeta values. This paper proves a conjecture of Kaneko predicting that an analogous formula holds for finite multiple zeta values.

1. Introduction

1.1. Finite multiple zeta values. The multiple zeta values (MZVs) and multiple zeta-star values (MZSVs) are defined by

\[ \zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}, \]

\[ \zeta^*(k_1, \ldots, k_n) = \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \]

for \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1} \) with \( k_1 \geq 2 \). They are both generalizations of the Riemann zeta values \( \zeta(k) \) at positive integers.

Among a large number of variants of the MZ(S)Vs, there has recently been growing interest in finite multiple zeta(-star) values (FMZ(S)Vs). Set \( A = (\prod_p \mathbb{Z}/p\mathbb{Z})/\bigoplus_p \mathbb{Z}/p\mathbb{Z} \), where \( p \) runs over all primes; in other words, the elements of \( A \) are of the form \( (a_p)_p \), where \( a_p \in \mathbb{Z}/p\mathbb{Z} \), and two elements \( (a_p) \) and \( (b_p) \) are identified if and only if \( a_p = b_p \) for all but finitely many primes \( p \). We shall simply write \( a_p \) for \( (a_p)_p \) since no confusion is likely. The following definition is due to Zagier (see [6]):

Definition 1.1. For \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1} \), we define

\[ \zeta_A(k_1, \ldots, k_n) = \sum_{p> m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \in A, \]

\[ \zeta_A^*(k_1, \ldots, k_n) = \sum_{p> m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \in A \]

and call them finite multiple zeta(-star) values.

We spell out two easy properties of FMZ(S)Vs that will be used later; see Theorems 4.3 and 6.1 in [3] for the proofs. See also [6, 8] and the introduction of [7].

Proposition 1.2. (1) We have \( \zeta_A(k) = 0 \) for all \( k \in \mathbb{Z}_{\geq 1} \).

(2) For \( k_1, k_2 \in \mathbb{Z}_{\geq 1} \), we have

\[ \zeta_A(k_1, k_2) = \zeta_A(k_1, k_2) = (-1)^{k_1} \binom{k_1 + k_2}{k_1} \frac{B_{p-k_1-k_2}}{k_1 + k_2}. \]

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Here the numbers $B_m$ are the Bernoulli numbers given by
\[
\sum_{m=0}^{\infty} B_m \frac{x^m}{m!} = \frac{x}{1 - e^{-x}} \in \mathbb{Q}[[x]].
\]

1.2. Sum formula. The sum formula is a basic class of relations among MZ(S)Vs and has been generalized in various directions. For $k, n \in \mathbb{Z}$ with $1 \leq n \leq k - 1$, set
\[
I_{k,n} = \{(k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 1}^n \mid k_1 + \cdots + k_n = k, \ k_1 \geq 2\}.
\]

**Theorem 1.3** (Sum formula [1, 2]). For $k, n \in \mathbb{Z}$ with $1 \leq n \leq k - 1$, we have
\[
\sum_{(k_1, \ldots, k_n) \in I_{k,n}} \zeta(k_1, \ldots, k_n) = \zeta(k),
\]
\[
\sum_{(k_1, \ldots, k_n) \in I_{k,n}} \zeta^*(k_1, \ldots, k_n) = \binom{k-1}{n-1} \zeta(k).
\]

Kaneko [5] conjectured the following analogous relations for FMZ(S)Vs:
\[
\sum_{(k_1, \ldots, k_n) \in I_{k,n}} \zeta_A(k_1, \ldots, k_n) = \left(1 + (-1)^n \frac{k-1}{n-1}\right) \frac{B_{p-k}}{k},
\]
\[
\sum_{(k_1, \ldots, k_n) \in I_{k,n}} \zeta^*_A(k_1, \ldots, k_n) = \left((-1)^n + \frac{k-1}{n-1}\right) \frac{B_{p-k}}{k}.
\]

The aim of this paper is to prove the conjecture and its generalizations given below.

For $k, n, i \in \mathbb{Z}$ with $1 \leq i \leq n \leq k - 1$, set
\[
I_{k,n,i} = \{(k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 1}^n \mid k_1 + \cdots + k_n = k, \ k_i \geq 2\};
\]

note that $I_{k,n,1} = I_{k,n}$.

**Theorem 1.4** (Main theorem). For $k, n, i \in \mathbb{Z}$ with $1 \leq i \leq n \leq k - 1$, we have
\[
\sum_{(k_1, \ldots, k_n) \in I_{k,n,i}} \zeta_A(k_1, \ldots, k_n) = (-1)^{i-1} \left(\frac{k-1}{i-1} + (-1)^n \frac{k-1}{n-i}\right) \frac{B_{p-k}}{k},
\]
\[
\sum_{(k_1, \ldots, k_n) \in I_{k,n,i}} \zeta^*_A(k_1, \ldots, k_n) = (-1)^{i-1} \left((-1)^n \frac{k-1}{i-1} + \frac{k-1}{n-i}\right) \frac{B_{p-k}}{k}.
\]

Setting $i = 1$ gives Kaneko’s conjecture.

2. Proof of the main theorem

For notational simplicity, we write the sums to be computed as
\[
S_{k,n,i} = \sum_{(k_1, \ldots, k_n) \in I_{k,n,i}} \zeta_A(k_1, \ldots, k_n), \quad S^*_{k,n,i} = \sum_{(k_1, \ldots, k_n) \in I_{k,n,i}} \zeta^*_A(k_1, \ldots, k_n)
\]

for $k, n, i \in \mathbb{Z}$ with $1 \leq i \leq n \leq k - 1$. 
2.1. Recurrence relations. We begin the proof by establishing recurrence relations for $S_{k,n,i}$ and $S_{k,n,i}^*$. We will show the recurrence relations by expressing products of FMZ(S)Vs as sums of FMZ(S)Vs via the harmonic product (see [4]). Since explaining the harmonic product in its full generality is unnecessarily cumbersome, we shall only illustrate it by examples. If $k_1, k_2, l \in \mathbb{Z}_{\geq 1}$, then Proposition 1.2 (1) shows that

$$0 = \zeta_A(k_1, k_2)\zeta_A(l)$$

$$= \left( \sum_{m_1 > m_2} + \sum_{m_1 > m_2} + \sum_{m_1 = m_2 > m} + \sum_{m_1 = m_2 = m} \right) \frac{1}{m_1^k m_2^l}$$

$$= \zeta_A(l, k_1, k_2) + \zeta_A(k_1, l, k_2) + \zeta_A(k_1, k_2, l) + \zeta_A(k_1 + l, k_2) + \zeta_A(k_1, k_2 + l),$$

where $m_1, m_2, m$ are all assumed to be positive integers less than $p$, and similarly that

$$0 = \zeta_A^*(k_1, k_2) + \zeta_A^*(k_1, l, k_2) + \zeta_A^*(k_1, k_2, l) - \zeta_A^*(k_1 + l, k_2) - \zeta_A^*(k_1, k_2 + l).$$

An analogous procedure leads to the following lemma:

**Lemma 2.1.** For $n \in \mathbb{Z}_{\geq 2}$ and $k_1, \ldots, k_{n-1}, l \in \mathbb{Z}_{\geq 1}$, we have

$$\sum_{j=1}^{n} \zeta_A(k_1, \ldots, k_{j-1}, l, k_j, \ldots, k_{n-1}) + \sum_{j=1}^{n-1} \zeta_A(k_1, \ldots, k_{j-1}, k_j + l, k_{j+1}, \ldots, k_{n-1}) = 0,$$

$$\sum_{j=1}^{n} \zeta_A^*(k_1, \ldots, k_{j-1}, l, k_j, \ldots, k_{n-1}) - \sum_{j=1}^{n-1} \zeta_A^*(k_1, \ldots, k_{j-1}, k_j + l, k_{j+1}, \ldots, k_{n-1}) = 0.$$

**Proof.** Expand the left-hand sides of $\zeta_A(k_1, \ldots, k_{n-1})\zeta_A(l)$ and $\zeta_A^*(k_1, \ldots, k_{n-1})\zeta_A^*(l) = 0$. □

**Proposition 2.2** (Recurrence relations). For $k, n, i \in \mathbb{Z}$ with $2 \leq i + 1 \leq n \leq k - 1$, we have

$$(n - i)S_{k,n,i} + iS_{k,n,i+1} + (k - n)S_{k,n-1,i} = 0,$$

$$(n - i)S_{k,n,i}^* + iS_{k,n,i+1}^* - (k - n)S_{k,n-1,i}^* = 0.$$

**Proof.** Summing the equations in Lemma 2.1 over all $(k_1, \ldots, k_{n-1}, l) \in I_{k,n,i}$ gives the desired recurrence relations. Indeed, the map

$$(k_1, \ldots, k_{n-1}, l) \mapsto (k_1, \ldots, k_{j-1}, l, k_j, \ldots, k_{n-1})$$

defined on $I_{k,n,i}$ is a bijection onto $I_{k,n,i+1}$ for $j = 1, \ldots, i$ and onto $I_{k,n,i}$ for $j = i + 1, \ldots, n$; under the map

$$((k_1, \ldots, k_{n-1}, l), j) \mapsto (k_1, \ldots, k_{j-1}, k_j + l, k_{j+1}, \ldots, k_{n-1})$$

from $I_{k,n,i} \times \{1, \ldots, n - 1\}$ to $I_{k,n-1,i}$, the preimage of each $(k_1', \ldots, k_{n-1}') \in I_{k,n-1,i}$ is of cardinality

$$\sum_{1 \leq j \leq n-1 \atop j \not= i} (k_j' - 1) + (k_i' - 2) = k - n.$$ □
2.2. Computation of $S^*_{k,n,i}$.

**Lemma 2.3** (Initial values). For $k, i \in \mathbb{Z}$ with $1 \leq i \leq k - 1$, we have

$$S^*_{k,k-1,i} = (-1)^{i-1} \binom{k}{i} B_{p-k}.$$  

**Proof.** By the duality theorem for FMZSVs [3, Theorem 4.6] and Proposition 1.2 (2), we find that

$$S^*_{k,k-1,i} = \zeta^\star_A(1, \ldots, 1, 2, 1, \ldots, 1) = -\zeta^\star_A(i, k-i) = (-1)^{i-1} \binom{k}{i} B_{p-k}.$$

\[\square\]

**Proposition 2.4.** For $k, n, i \in \mathbb{Z}$ with $1 \leq i \leq n \leq k - 1$, we have

$$S^*_{k,n,i} = (-1)^{i-1} \left( (-1)^n \binom{k-1}{i-1} + \binom{k-1}{n-i} \right) B_{p-k}.$$  

**Proof.** The proof is by backward induction on $n$.

We first consider the case $n = k - 1$. If $k$ is even, then the identity trivially follows from Lemma 2.3 because $B_{p-k} = 0$ (in $\mathbb{Q}$ and so in $\mathbb{Z}/p\mathbb{Z}$ as well) whenever $p$ is a prime at least $k + 3$. If $k$ is odd, then the identity again follows from Lemma 2.3 because

$$(-1)^n \binom{k-1}{i-1} + \binom{k-1}{n-i} = \binom{k-1}{i-1} + \binom{k-1}{k-i-1} = \binom{k}{i}.$$  

Now assume that the identity holds for $n$. Then Proposition 2.2 shows that

$$(k-n)S^*_{k,n-1,i} = (n-i)S^*_{k,n,i} + iS^*_{k,n,i+1} = (n-i)(-1)^{i-1} \left( (-1)^n \binom{k-1}{i-1} + \binom{k-1}{n-i} \right) B_{p-k}$$  

$$+ i(-1)^i \left( (-1)^n \binom{k-1}{i} + \binom{k-1}{n-i-1} \right) B_{p-k}$$  

$$= (-1)^{i-1} \left( (n-i)(-1)^n \binom{k-1}{i-1} + (k-n+i) \binom{k-1}{n-i-1} \right) B_{p-k}$$  

$$+ (-1)^i \left( (k-i)(-1)^n \binom{k-1}{i-1} + i \binom{k-1}{n-i-1} \right) B_{p-k}$$  

$$= (k-n)(-1)^{i-1} \left( (-1)^{n-1} \binom{k-1}{i-1} + \binom{k-1}{n-i-1} \right) B_{p-k}.$$  

Therefore the identity holds for $n + 1$ as well and the proof is complete.  

\[\square\]

2.3. Computation of $S_{k,n,i}$. Observe that each (F)MZV can be written as a $\mathbb{Z}$-linear combination of (F)MZSVs and vice versa, an example being

$$\zeta_A(k_1, k_2, k_3) = \sum_{m_1 > m_2 > m_3} \frac{1}{m_1^{k_1} m_2^{k_2} m_3^{k_3}}$$

$$= \left( \sum_{m_1 > m_2 > m_3} - \sum_{m_1 = m_2 > m_3} - \sum_{m_1 > m_2 = m_3} + \sum_{m_1 = m_2 = m_3} \right) \frac{1}{m_1^{k_1} m_2^{k_2} m_3^{k_3}}$$

$$= \zeta^\star_A(k_1, k_2, k_3) - \zeta^\star_A(k_1 + k_2, k_3) - \zeta^\star_A(k_1, k_2 + k_3) + \zeta^\star_A(k_1 + k_2 + k_3),$$  

where $m_1$, $m_2$, and $m_3$ are all assumed to be positive integers less than $p$. 
Lemma 2.5. For $k, n \in \mathbb{Z}$ with $1 \leq n \leq k - 1$, we have

$$S_{k,n,1} = \sum_{j=0}^{n-1} (-1)^j \binom{k-n+j-1}{j} S_{k,n-j,1}^*.$$ 

Proof. Each $\zeta_A(k_1, \ldots, k_n)$, where $(k_1, \ldots, k_n) \in I_{k,n,1}$, can be written as a sum of the values of the form $(-1)^j \zeta_A^j(k_1', \ldots, k_{n-j}')$ where $j = 0, \ldots, n-1$ and $(k_1', \ldots, k_{n-j}') \in I_{k,n-j,1}$. Moreover, each $(k_1', \ldots, k_{n-j}') \in I_{k,n-j,1}$ appears in this manner exactly as many times as there are ways of adding $j$ bars to the $n-j-1$ existing bars in the gaps in the following sequence of stars, in such a way that no bar separates the leftmost two stars and no two bars are in the same gap:

\[
\begin{array}{ccccccccc}
\ast & \ast & \cdots & \ast & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
& & & & & \downarrow& & & & \\
& & & & & k_1' & & & & k_{n-j}'
\end{array}
\]

Since there are $(k_1' - 2) + (k_2' - 1) + \cdots + (k_{n-j}' - 1) = k - n + j - 1$ gaps that accept bars, the number of ways is $\binom{k-n+j-1}{j}$.

Lemma 2.6 (Initial values). For $k, n \in \mathbb{Z}$ with $1 \leq n \leq k - 1$, we have

$$S_{k,n,1} = \left(1 + (-1)^n \binom{k-1}{n-1}\right) \frac{B_{p-k}}{k}.$$ 

Proof. By Proposition 2.4 and Lemma 2.5, we have

$$S_{k,n,1} = \sum_{j=0}^{n-1} (-1)^j \binom{k-n+j-1}{j} (-1)^{n-j} + \binom{k-1}{n-j-1} \frac{B_{p-k}}{k}$$

$$= \left(1 + (-1)^n \sum_{j=0}^{n-1} \binom{k-n+j-1}{j} + \sum_{j=0}^{n-1} (-1)^j \binom{k-n+j-1}{j} \binom{k-1}{n-j-1} \right) \frac{B_{p-k}}{k}.$$ 

Recall that $(1-x)^{-m} = \sum_{j=0}^{\infty} \binom{m+j-1}{j} x^j \in \mathbb{Q}[[x]]$ for $m \in \mathbb{Z}_{\geq 1}$. Looking at the coefficient of $x^{n-1}$ in the product of $(1-x)^{-(k-n)}$ and $(1-x)^{-1}$ gives

$$\sum_{j=0}^{n-1} \binom{k-n+j-1}{j} = \binom{(k-n+1)+(n-1)-1}{n-1} = \binom{k-1}{n-1};$$

looking at the coefficient of $x^{n-1}$ in the product of $(1+x)^{-(k-n)}$ and $(1+x)^{k-1}$ gives

$$\sum_{j=0}^{n-1} (-1)^j \binom{k-n+j-1}{j} \binom{k-1}{n-j-1} = 1.$$ 

The proof is now complete.

Proposition 2.7. For $k, n, i \in \mathbb{Z}$ with $1 \leq i \leq n \leq k - 1$, we have

$$S_{k,n,i} = (-1)^{i-1} \left( \binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} \right) \frac{B_{p-k}}{k}.$$
Proof. The proof is by induction on $i$, the case $i = 1$ being Lemma 2.6. Assume that the identity holds for $i$. Then Proposition 2.2 shows that

$$i S_{k,n,i+1} = - (n - i) S_{k,n,i} - (k - n) S_{k,n-1,i}$$

$$= - (n - i) (-1)^{i-1} \binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} \frac{B_{p-k}}{k}$$

$$- (k - n) (-1)^{i-1} \binom{k-1}{i-1} + (-1)^{n-1} \binom{k-1}{n-i-1} \frac{B_{p-k}}{k}$$

$$= - (-1)^{i-1} \binom{n - i}{i-1} + (k - n - i) (-1)^{n} \binom{k-1}{n-i-1} \frac{B_{p-k}}{k}$$

$$- (-1)^{i-1} \binom{k-1}{i-1} + (k - n) (-1)^{n-1} \binom{k-1}{n-i-1} \frac{B_{p-k}}{k}$$

$$= (-1)^i \binom{k-1}{i} \binom{k-1}{i-1} + i (-1)^n \binom{k-1}{n-i-1} \frac{B_{p-k}}{k}$$

$$= i (-1)^i \binom{k-1}{i} + (-1)^n \binom{k-1}{n-i-1} \frac{B_{p-k}}{k}.$$ 

Therefore the identity holds for $i + 1$ as well and the proof is complete. \qed

Combining Propositions 2.4 and 2.7, we have completed the proof of the main theorem (Theorem 1.4).

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