Outline proof of the equivalence concerning knot points of typical continuous functions

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Let \( I = [0, 1] \) and look at the Banach space \( C(I) = \{ f: I \to \mathbb{R} \mid f \text{ is continuous} \} \).

**Definition 1.**
We say that a typical (generic) \( f \in C(I) \) has property \( P \) if the set \( \{ f \in C(I) \mid f \text{ has property } P \} \) is residual.

Recall that a subset \( A \) of a topological space is said to be nowhere dense if the closure of \( A \) has empty interior; \( A \) is meagre (first category) if \( A \) can be expressed as a countable union of nowhere dense sets; \( A \) is residual (comeagre) if its complement \( A^c \) is meagre.

The investigation of the behaviour of typical functions \( f \in C(I) \) started when Banach [Ba] and Mazurkiewicz [Ma] independently proved in 1931 that a typical \( f \in C(I) \) is nowhere differentiable. The theorem means that we need to consider Dini derivatives rather than ordinary derivatives for typical functions.

**Definition 2.**
The Dini derivatives of \( f \in C(I) \) at \( x \in I \) are the extended real numbers defined by

\[
D^+ f(x) = \limsup_{h \downarrow 0} \frac{f(x + h) - f(x)}{h}, \quad D^- f(x) = \limsup_{h \uparrow 0} \frac{f(x + h) - f(x)}{h},
\]
\[
D_+ f(x) = \liminf_{h \downarrow 0} \frac{f(x + h) - f(x)}{h}, \quad D_- f(x) = \liminf_{h \uparrow 0} \frac{f(x + h) - f(x)}{h}.
\]

At endpoints of \( I \), we may define only two of the Dini derivatives: \( D^+ f(0) \) and \( D^- f(0) \) at 0, and \( D^- f(1) \) and \( D_+ f(1) \) at 1.

Jarník [Ja] proved the following theorem concerning Dini derivatives of typical functions:

**Theorem 3 (Jarník).**
A typical \( f \in C(I) \) has the property that

\[
D^+ f(x) = D^- f(x) = \infty, \quad D_+ f(x) = D_- f(x) = -\infty
\]

at almost all \( x \in I \).

The function \( f \) may be considered to be the least differentiable at such a point \( x \).
Definition 4.
Let $f \in C(I)$. A point $x \in I$ is called a knot point of $f$ if

$$D^+ f(x) = D^- f(x) = \infty, \quad D_+ f(x) = D_- f(x) = -\infty.$$ 

We write $N(f)$ for the set of all points in $I$ that are not knot points of $f$.

An endpoint of $I$ is called a knot point if the two Dini derivatives defined there are $\infty$ and $-\infty$. For example, $0 \in I$ is a knot point of $f \in C(I)$ if $D^+ f(0) = \infty$ and $D_+ f(0) = -\infty$.

Jarník’s theorem is equivalent to saying that $N(f)$ is null for a typical $f \in C(I)$. A natural question generalising the theorem is in what sense $N(f)$ is small for a typical $f \in C(I)$. The question has been completely answered by Preiss and Zajíček [PZ]. To state their theorem, we write $\mathcal{K}$ for the family of all closed (or equivalently compact) subsets of $I$, and equip $\mathcal{K}$ with the Hausdorff metric. It is known that the Hausdorff metric makes $\mathcal{K}$ a compact metric space (see [Ke, Theorem 4.26]).

Theorem 5 (Preiss & Zajíček, unpublished).
For a $\sigma$-ideal $\mathcal{I}$ on $I$, the following are equivalent:

1. a typical $f \in C(I)$ has the property that $N(f) \in \mathcal{I}$;
2. a typical $K \in \mathcal{K}$ belongs to $\mathcal{I}$ (i.e. $I \cap K$ is a residual subset of $\mathcal{K}$).

Recall that a $\sigma$-ideal on $I$ is a nonempty family $\mathcal{I}$ of subsets of $I$ with the following properties:

- if $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$;
- if $A_n \in \mathcal{I}$ for $n \in \mathbb{N}$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{I}$.

A $\sigma$-ideal on $I$ can be regarded as a family of small subsets of $I$.

Now we shall extend Theorem 5 to general families of subsets of $I$ rather than $\sigma$-ideals. That is to say, given an arbitrary family $\mathcal{S}$ of subsets of $I$, we seek a method for deciding whether $N(f) \in \mathcal{S}$ for a typical $f \in C(I)$. Observing that $N(f)$ is always an $F_{\sigma}$ set (countable union of closed sets), we only need to look at families of $F_{\sigma}$ subsets of $I$. The following is the main theorem of this article:

Theorem 6 (Preiss & S.).
For a family $\mathcal{F}$ of $F_{\sigma}$ subsets of $I$, the following are equivalent:

1. a typical $f \in C(I)$ has the property that $N(f) \in \mathcal{F}$;
2. a typical $(K_n) \in \mathcal{K}^\infty$ has the property that $\bigcup_{n=1}^\infty K_n \in \mathcal{F}$.

Here $\mathcal{K}^\infty$ is the countable product of $\mathcal{K}$, equipped with the product topology.

Below we shall give an outline proof of Theorem 6. A complete proof will appear in a joint paper [PS], which is still in preparation, but is available in the author’s PhD thesis [Sa].

Theorem 6 reduces to constructing such $\mathcal{X}$ as in the following lemma:

Lemma 7.
There exists $\mathcal{X} \subset \mathcal{K}^\infty \times C(I)$ with the following properties:
(A) if \((K_n), f \in \mathcal{X}\), then \(\bigcup_{n=1}^{\infty} K_n = N(f)\);
(B) if \(A \subset \mathcal{K}^\mathbb{N}\) is residual, then a typical \(f \in C(I)\) has the property that \((K_n), f \in \mathcal{X}\) for some \((K_n) \in A\);
(C) \(\mathcal{X}\) is analytic;
(D) for every \(f \in C(I)\), the set \(\{ (K_n) \in \mathcal{K}^\mathbb{N} \mid (K_n), f \in \mathcal{X} \}\) is closed under finite permutations.

Recall that a **Polish space** is a completely metrisable separable topological space; a subset \(A\) of a Polish space \(X\) is said to be **analytic** if there exist a Polish space \(Y\) and a Borel subset \(B\) of \(X \times Y\) such that the first projection of \(B\) is \(A\). We say that a subset \(A\) of \(\mathcal{K}^\mathbb{N}\) is **closed under finite permutations** if \((K_{\sigma(n)}) \in A\) whenever \((K_n) \in A\) and \(\sigma\) is a permutation on \(\mathbb{N}\) for which \(\{ n \in \mathbb{N} \mid \sigma(n) \neq n \}\) is a finite set.

The proof of Lemma 7 relies on constructing \(\mathcal{X}\) concretely and showing that it does indeed have properties (A)–(D). In verifying property (B), we construct a winning strategy for a Banach-Mazur game on \(C(I)\) (see [Ke, Section 8.H] for the Banach-Mazur game).

In what follows we prove Theorem 6 assuming Lemma 7. The implication \(\text{(2) } \Rightarrow \text{(1)}\) is easy:

**Proof of \(\text{(2) } \Rightarrow \text{(1)}\) in Theorem 6.**

Set \(A = \{ (K_n) \in \mathcal{K}^\mathbb{N} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \}\). Since \(A\) is residual by assumption, Lemma 7 (B) shows that a typical \(f \in C(I)\) has the property that \((K_n), f \in \mathcal{X}\) for some \((K_n) \in A\). For such \(f\), the definition of \(A\) and Lemma 7 (B) give \(N(f) = \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\), verifying (1).

For the proof of the converse, we invoke two results in descriptive set theory:

**Lemma 8 ([Ke, Theorem 21.6]).**

Every analytic subset of a Polish space has the Baire property; namely it can be expressed as the symmetric difference of an open set and a meagre set.

**Lemma 9 (Topological zero-one law).**

If \(A \subset \mathcal{K}^\mathbb{N}\) is closed under finite permutations and has the Baire property, then it is either meagre or residual.

**Proof of \(\text{(1) } \Rightarrow \text{(2)}\) in Theorem 6.**

Since \(\{ f \in C(I) \mid N(f) \in \mathcal{F} \}\) is residual by assumption, it contains a dense \(G_\delta\) set (countable intersection of open sets) \(G\). Setting

\[ A = \{ (K_n) \in \mathcal{K}^\mathbb{N} \mid (K_n), f \in \mathcal{X} \text{ for some } f \in G \}, \]

the definition of \(G\) and Lemma 7 (A) show that if \((K_n) \in A\), then \(\bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\); so it suffices to show that \(A\) is residual.

Since \(A = \bigcup_{f \in G} \{ (K_n) \in \mathcal{K}^\mathbb{N} \mid (K_n), f \in \mathcal{X} \}\), it is closed under finite permutations by Lemma 7 (D). Moreover, since \(A\) is the first projection of \(\mathcal{X} \cap (\mathcal{K}^\mathbb{N} \times G)\), it is analytic by Lemma 7 (C) and so has the Baire property by Lemma 8. It follows from Lemma 9 that \(A\) is either meagre or residual.
We look for a contradiction assuming that $\mathcal{A}$ is meagre. Then since $\mathcal{A}^c$ is residual, Lemma 7 (B) shows that a typical $f \in C(I)$ has the property that $((K_n), f) \in \mathcal{E}$ for some $(K_n) \in \mathcal{A}^c$. This, together with the residuality of $G$, implies that $((K_n), f) \in \mathcal{E}$ for some $f \in G$ and $(K_n) \in \mathcal{A}^c$, which contradicts the definition of $\mathcal{A}$.

References


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