Outline proof of the equivalence concerning knot points of typical continuous functions

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Let $I = [0, 1]$ and look at the Banach space $C(I) = \{f : I \longrightarrow \mathbb{R} \mid f \text{ is continuous}\}$.

Definition 1.
We say that a **typical** (**generic**) $f \in C(I)$ has property P if the set $\{f \in C(I) \mid f \text{ has property } P\}$ is residual.

Recall that a subset A of a topological space is said to be **nowhere dense** if the closure of A has empty interior; A is **meagre** (**first category**) if A can be expressed as a countable union of nowhere dense sets; A is **residual** (**comeagre**) if its complement A^c is meagre.

The investigation of the behaviour of typical functions $f \in C(I)$ started when Banach [Ba] and Mazurkiewicz [Ma] independently proved in 1931 that a typical $f \in C(I)$ is nowhere differentiable. The theorem means that we need to consider Dini derivatives rather than ordinary derivatives for typical functions.

Definition 2.
The **Dini derivatives** of $f \in C(I)$ at $x \in I$ are the extended real numbers defined by

$$D^+f(x) = \limsup_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}, \qquad D^-f(x) = \limsup_{h \uparrow 0} \frac{f(x+h) - f(x)}{h},$$
$$D_+f(x) = \liminf_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}, \qquad D_-f(x) = \liminf_{h \uparrow 0} \frac{f(x+h) - f(x)}{h}.$$

At endpoints of I, we may define only two of the Dini derivatives: $D^+f(0)$ and $D_+f(0)$ at 0, and $D^-f(1)$ and $D_-f(1)$ at 1.

Jarník [Ja] proved the following theorem concerning Dini derivatives of typical functions:

Theorem 3 (Jarník).
A typical $f \in C(I)$ has the property that

$$D^+f(x) = D^-f(x) = \infty, \qquad D_+f(x) = D_-f(x) = -\infty$$

at almost all $x \in I$.

The function f may be considered to be the *least differentiable* at such a point x.

Definition 4.
Let $f \in C(I)$. A point $x \in I$ is called a **knot point** of f if

$$D^+ f(x) = D^- f(x) = \infty, \qquad D_+ f(x) = D_- f(x) = -\infty.$$

We write $N(f)$ for the set of all points in I that are *not* knot points of f.

An endpoint of I is called a knot point if the two Dini derivatives defined there are ∞ and $-\infty$. For example, $0 \in I$ is a knot point of $f \in C(I)$ if $D^+ f(0) = \infty$ and $D_+ f(0) = -\infty$.

Jarník's theorem is equivalent to saying that $N(f)$ is null for a typical $f \in C(I)$. A natural question generalising the theorem is in what sense $N(f)$ is small for a typical $f \in C(I)$. The question has been completely answered by Preiss and Zajíček [PZ]. To state their theorem, we write $\mathcal{K}$ for the family of all closed (or equivalently compact) subsets of I, and equip $\mathcal{K}$ with the Hausdorff metric. It is known that the Hausdorff metric makes $\mathcal{K}$ a compact metric space (see [Ke, Theorem 4.26]).

Theorem 5 (Preiss & Zajíček, unpublished).
For a σ-ideal $\mathcal{I}$ on I, the following are equivalent:
(1) *a typical $f \in C(I)$ has the property that $N(f) \in \mathcal{I}$;*
(2) *a typical $K \in \mathcal{K}$ belongs to $\mathcal{I}$ (i.e. $\mathcal{I} \cap \mathcal{K}$ is a residual subset of $\mathcal{K}$).*

Recall that a ***σ*-ideal** on I is a nonempty family $\mathcal{I}$ of subsets of I with the following properties:
- if $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$;
- if $A_n \in \mathcal{I}$ for $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{I}$.

A σ-ideal on I can be regarded as a family of *small* subsets of I.

Now we shall extend Theorem 5 to general families of subsets of I rather than σ-ideals. That is to say, given an arbitrary family $\mathcal{S}$ of subsets of I, we seek a method for deciding whether $N(f) \in \mathcal{S}$ for a typical $f \in C(I)$. Observing that $N(f)$ is always an F_σ set (countable union of closed sets), we only need to look at families of F_σ subsets of I. The following is the main theorem of this article:

Theorem 6 (Preiss & S.).
For a family $\mathcal{F}$ of F_σ subsets of I, the following are equivalent:
(1) *a typical $f \in C(I)$ has the property that $N(f) \in \mathcal{F}$;*
(2) *a typical $(K_n) \in \mathcal{K}^{\mathbb{N}}$ has the property that $\bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$.*

Here $\mathcal{K}^{\mathbb{N}}$ is the countable product of $\mathcal{K}$, equipped with the product topology.

Below we shall give an outline proof of Theorem 6. A complete proof will appear in a joint paper [PS], which is still in preparation, but is available in the author's PhD thesis [Sa].

Theorem 6 reduces to constructing such $\mathscr{X}$ as in the following lemma:

Lemma 7.
There exists $\mathscr{X} \subset \mathcal{K}^{\mathbb{N}} \times C(I)$ with the following properties:

(A) *if* $\big((K_n), f\big) \in \mathscr{X}$, *then* $\bigcup_{n=1}^{\infty} K_n = N(f)$;

(B) *if* $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}$ *is residual, then a typical* $f \in C(I)$ *has the property that* $\big((K_n), f\big) \in \mathscr{X}$ *for some* $(K_n) \in \mathcal{A}$;

(C) $\mathscr{X}$ *is analytic*;

(D) *for every* $f \in C(I)$, *the set* $\big\{(K_n) \in \mathcal{K}^{\mathbb{N}} \bigm| \big((K_n), f\big) \in \mathscr{X}\big\}$ *is closed under finite permutations.*

Recall that a **Polish space** is a completely metrisable separable topological space; a subset A of a Polish space X is said to be **analytic** if there exist a Polish space Y and a Borel subset B of $X \times Y$ such that the first projection of B is A. We say that a subset $\mathcal{A}$ of $\mathcal{K}^{\mathbb{N}}$ is **closed under finite permutations** if $(K_{\sigma(n)}) \in \mathcal{A}$ whenever $(K_n) \in \mathcal{A}$ and σ is a permutation on $\mathbb{N}$ for which $\{n \in \mathbb{N} \mid \sigma(n) \neq n\}$ is a finite set.

The proof of Lemma 7 relies on constructing $\mathscr{X}$ concretely and showing that it does indeed have properties (A)–(D). In verifying property (B), we construct a winning strategy for a Banach-Mazur game on $C(I)$ (see [Ke, Section 8.H] for the Banach-Mazur game).

In what follows we prove Theorem 6 assuming Lemma 7. The implication (2) $\implies$ (1) is easy:

Proof of (2) $\implies$ (1) in Theorem 6.

Set $\mathcal{A} = \{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$. Since $\mathcal{A}$ is residual by assumption, Lemma 7 (B) shows that a typical $f \in C(I)$ has the property that $\big((K_n), f\big) \in \mathscr{X}$ for some $(K_n) \in \mathcal{A}$. For such f, the definition of $\mathcal{A}$ and Lemma 7 (B) give $N(f) = \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$, verifying (1). ∎

For the proof of the converse, we invoke two results in descriptive set theory:

Lemma 8 ([Ke, Theorem 21.6]).

Every analytic subset of a Polish space has the Baire property; *namely it can be expressed as the symmetric difference of an open set and a meagre set.*

Lemma 9 (Topological zero-one law).

If $\mathcal{A} \subset \mathcal{K}^{\mathbb{N}}$ *is closed under finite permutations and has the Baire property, then it is either meagre or residual.*

Proof of (1) $\implies$ (2) in Theorem 6.

Since $\{f \in C(I) \mid N(f) \in \mathcal{F}\}$ is residual by assumption, it contains a dense G_δ set (countable intersection of open sets) G. Setting

$$\mathcal{A} = \big\{(K_n) \in \mathcal{K}^{\mathbb{N}} \bigm| \big((K_n), f\big) \in \mathscr{X} \text{ for some } f \in G\big\},$$

the definition of G and Lemma 7 (A) show that if $(K_n) \in \mathcal{A}$, then $\bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$; so it suffices to show that $\mathcal{A}$ is residual.

Since $\mathcal{A} = \bigcup_{f \in G} \big\{(K_n) \in \mathcal{K}^{\mathbb{N}} \bigm| \big((K_n), f\big) \in \mathscr{X}\big\}$, it is closed under finite permutations by Lemma 7 (D). Moreover, since $\mathcal{A}$ is the first projection of $\mathscr{X} \cap (\mathcal{K}^{\mathbb{N}} \times G)$, it is analytic by Lemma 7 (C) and so has the Baire property by Lemma 8. It follows from Lemma 9 that $\mathcal{A}$ is either meagre or residual.

We look for a contradiction assuming that $\mathcal{A}$ is meagre. Then since $\mathcal{A}^c$ is residual, Lemma 7 (B) shows that a typical $f \in C(I)$ has the property that $\bigl((K_n), f\bigr) \in \mathscr{X}$ for some $(K_n) \in \mathcal{A}^c$. This, together with the residuality of G, implies that $\bigl((K_n), f\bigr) \in \mathscr{X}$ for some $f \in G$ and $(K_n) \in \mathcal{A}^c$, which contradicts the definition of $\mathcal{A}$. ∎

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