# The Erdös-Sierpiński Duality Theorem

Shingo SAITO*

## 0 Notation

The Lebesgue measure on $\mathbb{R}$ is denoted by $\mu$. The $\sigma$-ideals that consist of all meagre subsets and null subsets of $\mathbb{R}$ are denoted by $\mathcal{M}$ and $\mathcal{N}$ respectively.

## 1 Similarities between Meagre Sets and Null Sets

**Definition 1.1** Let $\mathcal{I}$ be an ideal on a set. A subset $\mathcal{B}$ of $\mathcal{I}$ is called a *base* for $\mathcal{I}$ if each set in $\mathcal{I}$ is contained in some set in $\mathcal{B}$.

**Proposition 1.2** *Each meagre subset of a topological space is contained in some meagre $\mathcal{F}_\sigma$ set. In particular, $\mathcal{M} \cap \mathcal{F}_\sigma$ is a base for $\mathcal{M}$.*

**Proof.** Let $A$ be a meagre subset of a topological space. Then $A = \bigcup_{n=1}^\infty A_n$ for some nowhere dense sets $A_n$. The set $\bigcup_{n=1}^\infty \overline{A_n}$, which contains $A$, is meagre and $\mathcal{F}_\sigma$ since the sets $\overline{A_n}$ are nowhere dense and closed. ∎

**Proposition 1.3** *Each null subset of $\mathbb{R}$ is contained in some null $\mathcal{G}_\delta$ set. In other words, $\mathcal{N} \cap \mathcal{G}_\delta$ is a base for $\mathcal{N}$.*

**Proof.** This is immediate from the regularity of the Lebesgue measure. ∎

**Proposition 1.4** *Every uncountable $\mathcal{G}_\delta$ subset of $\mathbb{R}$ contains a nowhere dense null closed set with cardinality $2^\omega$.*

**Proof.** Let $G$ be an uncountable $\mathcal{G}_\delta$ set. Then $G = \bigcap_{n=0}^\infty U_n$ for some open sets $U_n$.

We may construct a Cantor scheme $\{ I_s \mid s \in 2^{<\omega} \}$ such that $I_s$ is a compact nondegenerate interval contained in $U_{|s|}$ with $\mu(I_s) \leqq 3^{-n}$ for every $s \in 2^{<\omega}$. Let $f\colon 2^\omega \longrightarrow \mathbb{R}$ denote the associated map of the Cantor scheme defined by $\{f(\alpha)\} = \bigcap_{n=1}^\infty I_{\alpha|_n}$. Denote the range of $f$ by $A$.

Note that $A = \bigcap_{n=1}^\infty \bigcup_{s \in 2^n} I_s$, which implies that $A$ is closed. Moreover $A$ is null because

$$\mu(A) \leqq \mu\left(\bigcup_{s\in 2^n} I_s\right) \leqq \sum_{s \in 2^n} \mu(I_s) \leqq \frac{2^n}{3^n} \to 0 \quad \text{as } n \to \infty.$$

*new@ms.u-tokyo.ac.jp

Thus $\operatorname{Int} A = \emptyset$, which shows that $A$ is nowhere dense since $A$ is closed. The injectivity of $f$ shows that $|A| = 2^\omega$. Since $I_s \subset U_{|s|}$ for every $s \in 2^{<\omega}$, we have

$$A = \bigcap_{n=1}^{\infty} \bigcup_{s \in 2^n} I_s \subset \bigcap_{n=1}^{\infty} U_n = G.$$

∎

**Corollary 1.5** *Every residual subset of $\mathbb{R}$ contains a meagre set with cardinality $2^\omega$.*

**Proof.** Let $A$ be a residual subset of $\mathbb{R}$. Then $A^c = \bigcup_{n=1}^{\infty} A_n$ for some nowhere dense sets $A_n$. Since $\bigcup_{n=1}^{\infty} \overline{A}_n$ is a meagre $\mathcal{F}_\sigma$ set that contains $A^c$, the set $(\bigcup_{n=1}^{\infty} \overline{A}_n)^c$ is a nonmeagre $\mathcal{G}_\delta$ set contained in $A$. Therefore Proposition 1.4 shows that $(\bigcup_{n=1}^{\infty} \overline{A}_n)^c$ contains a meagre set with cardinality $2^\omega$, which is contained in $A$. ∎

**Corollary 1.6** *The complement of each null subset of $\mathbb{R}$ contains a null set with cardinality $2^\omega$.*

**Proof.** Let $A$ be a subset of $\mathbb{R}$ with $A^c \in \mathcal{N}$. Then $A$ is measurable and $\mu(A) = \infty$. Therefore the regularity of $\mu$ yields a closed set $F$ contained in $A$ with $\mu(F) = \infty$. It follows from Proposition 1.4 that $F$ contains a null set with cardinality $2^\omega$, which is contained in $A$. ∎

# 2 Erdös-Sierpiński Duality Theorem

**Proposition 2.1** *There exist a meagre $\mathcal{F}_\sigma$ subset $A$ and a null $\mathcal{G}_\delta$ subset $B$ of $\mathbb{R}$ that satisfy $A \cap B = \emptyset$ and $A \cup B = \mathbb{R}$.*

**Proof.** Enumerate $\mathbb{Q} = \{q_1, q_2, \ldots\}$, and put $P_n = \bigcup_{j=1}^{\infty} B(q_j, 2^{-n-j})$ for positive integers $n$. Define $B = \bigcap_{n=1}^{\infty} P_n$ and $A = B^c$. Since $P_n$ is open for every positive integer $n$, we have $B \in \mathcal{G}_\delta$ and $A \in \mathcal{F}_\sigma$. We shall prove that $A$ is meagre and $B$ is null.

The set $B$ is null because

$$\begin{aligned} \mu(B) \leqq \mu(P_n) \leqq \sum_{j=1}^{\infty} \mu\big(B(q_j, 2^{-n-j})\big) &= \sum_{j=1}^{\infty} 2^{-n-j+1} \\ = 2^{-n+1} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

For every positive integer $n$, the set $P_n^c$ is nowhere dense because $P_n$ is open and dense. It follows from $A = \bigcup_{n=1}^{\infty} P_n^c$ that $A$ is meagre. ∎

**Theorem 2.2 (Erdös-Sierpiński Duality Theorem)** *Assume that the continuum hypothesis holds. Then there exists an involution $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(A)$ is meagre if and only if $A$ is null, and $f(A)$ is null if and only if $A$ is meagre for every subset $A$ of $\mathbb{R}$.*

**Proof.** Since $|\mathcal{M} \cap \mathcal{F}_\sigma| = 2^\omega$ and $|\mathcal{N} \cap \mathcal{G}_\delta| = 2^\omega$, it follows from Proposition 2.1 that there exist bijections $\xi \longmapsto A_\xi$ from $2^\omega$ to $\mathcal{M} \cap \mathcal{F}_\sigma$, and $\xi \longmapsto B_\xi$ from $2^\omega$ to $\mathcal{N} \cap \mathcal{G}_\delta$ that satisfy $A_0 \cap B_0 = \emptyset$ and $A_0 \cup B_0 = \mathbb{R}$.

Define inductively a map $\xi \longmapsto F_\xi$ from $2^\omega$ to $\mathcal{M}$ such that

(1) $F_0 = A_0$;

(2) $F_{\xi+1}$ is the union of $F_\xi \cup A_\xi$ and a meagre set contained in $(F_\xi \cup A_\xi)^c$ with cardinality $2^\omega$ for every $\xi \in 2^\omega$;

(3) $F_\xi = \bigcup_{\alpha \in \xi} F_\alpha$ for every limit ordinal $\xi \in 2^\omega$.

We may construct such a map due to Corollary 1.5 and the continuum hypothesis.

Similarly Corollary 1.6 and the continuum hypothesis allow us to define a map $\xi \longmapsto G_\xi$ from $2^\omega$ to $\mathcal{N}$ such that

(1) $G_0 = B_0$;

(2) $G_{\xi+1}$ is the union of $G_\xi \cup B_\xi$ and a null set contained in $(G_\xi \cup B_\xi)^c$ with cardinality $2^\omega$ for every $\xi \in 2^\omega$;

(3) $G_\xi = \bigcup_{\alpha \in \xi} G_\alpha$ for every limit ordinal $\xi \in 2^\omega$.

For each $\xi \in 2^\omega$, there exists a bijection $f_\xi \colon F_{\xi+1} \setminus F_\xi \longrightarrow G_{\xi+1} \setminus G_\xi$ since $|F_{\xi+1} \setminus F_\xi| = |G_{\xi+1} \setminus G_\xi| = 2^\omega$. Put $\widetilde{f} = \bigcup_{\xi \in 2^\omega} f_\xi$. Then $\widetilde{f}$ is a bijection from $\bigcup_{\xi \in 2^\omega} (F_{\xi+1} \setminus F_\xi) = \mathbb{R} \setminus F_0 = G_0$ to $\bigcup_{\xi \in 2^\omega} (G_{\xi+1} \setminus G_\xi) = \mathbb{R} \setminus G_0 = F_0$. Thus the union $f$ of $\widetilde{f}$ and $\widetilde{f}^{-1}$ is an involution from $\mathbb{R}$ to $\mathbb{R}$.

Let $M$ be a meagre set. Proposition 1.2 implies that $M$ is contained in $A_{\xi_0}$ for some $\xi_0 \in 2^\omega$. Then

$$M \subset A_{\xi_0} \subset F_{\xi_0+1} = \bigcup_{\xi \in \xi_0+1} (F_{\xi+1} \setminus F_\xi) \cup F_0$$

shows that

$$\begin{aligned} f(M) \subset f\left( \bigcup_{\xi \in \xi_0+1} (F_{\xi+1} \setminus F_\xi) \cup F_0 \right) &= \bigcup_{\xi \in \xi_0+1} f(F_{\xi+1} \setminus F_\xi) \cup f(F_0) \\ &= \bigcup_{\xi \in \xi_0+1} (G_{\xi+1} \setminus G_\xi) \cup G_0 = G_{\xi_0+1} \in \mathcal{N}, \end{aligned}$$

which implies that $f(M)$ is null.

Similarly it follows from Proposition 1.3 that $f(N)$ is meagre for every null set $N$.

Since $f$ is an involution, we conclude that $f(M)$ is null only if $M$ is meagre, and that $f(N)$ is meagre only if $N$ is null. ∎

**Remark.** Assuming the continuum hypothesis is too much; the proof of Theorem 2.2 works on the mere assumption that $\operatorname{add}(\mathcal{M}) = \operatorname{add}(\mathcal{N}) = 2^\omega$. In particular, assuming the Martin axiom is enough.

# 3 References

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